

# Coloring Random Triangulations

P. Di Francesco\*,

*Department of Mathematics,  
University of North Carolina at Chapel Hill,  
CHAPEL HILL, N.C. 27599-3250, U.S.A.*

B. Eynard<sup>#</sup>

*Department of Mathematical Sciences,  
University of Durham, Science Labs,  
South Road, DURHAM DH1 3HP, U.K.*

E. Guitter<sup>\$</sup>

*Service de Physique Théorique,  
C.E.A. Saclay,  
F-91191 Gif sur Yvette Cedex, France*

We introduce and solve a two-matrix model for the tri-coloring problem of the vertices of a random triangulation. We present three different solutions: (i) by orthogonal polynomial techniques (ii) by use of a discrete Hirota bilinear equation (iii) by direct expansion. The model is found to lie in the universality class of pure two-dimensional quantum gravity, despite the non-polynomiality of its potential.

P.A.C.S. numbers: 05.20.y, 04.60.Nc – Keywords: Coloring, Folding, Random Lattice, 2D Quantum Gravity

---

\* e-mail: philippe@math.unc.edu

<sup>#</sup> e-mail: Bertrand.Eynard@durham.ac.uk

<sup>\$</sup> e-mail: guitter@spht.saclay.cea.fr

## 1. Introduction

Various graph coloring problems have been considered in the last two decades in relation with statistical mechanics. On regular lattices, many of these have proved to be integrable and exactly solvable [1]. Among these is the classical tri-coloring problem of the bonds of the triangular lattice, solved by Baxter in [2]: in how many ways can one color the lattice bonds with three colors so that any triangle has its three edges of a different color. Recently a remarkable connection was made between this problem and that of folding of the triangular lattice [3], counting the different ways to fold the lattice onto itself along its bonds, a toy model for the crumpling of two-dimensional polymerized membranes. A natural extension of these problems consists in their generalization to random triangulations. This may be viewed as a coupling of the model to two-dimensional quantum gravity, or also as a way to describe fluid membranes. The bond tri-coloring of random triangulations has first been addressed in [4] and solved in [5] for a restricted model where the colors always occur in the same cyclic order for all triangles. The complete bond tri-coloring problem without restriction was solved very recently in [6]. The connection to the folding of random triangulations requires unfortunately an additional constraint: the number of triangles around each vertex must be even. The folding of random triangulations remains an open problem.

In this paper, we address a different kind of (bi- or tri-)coloring problems, where we now color the vertices or the faces of some random graphs, representing tessellated surfaces of arbitrary genus. Starting from the vertex bi-coloring of arbitrary graphs, our main result will eventually be rephrased as solving the tri-coloring problem of the vertices of random triangulations of surfaces.

Our starting point is the computation of the following integral over two Hermitian matrices  $M$  and  $R$  of size  $N \times N$  (see [7] for a review)

$$Z(p, q, g; N) = \int dM dR e^{-N \text{Tr} V(M, R; p, q, g)} \quad (1.1)$$

with the non-polynomial potential

$$V(M, R; p, q, g) = p \text{Log}(1 - M) + q \text{Log}(1 - R) + gMR \quad (1.2)$$

and with respect to the Haar measure over Hermitian matrices  $H$

$$dH = c_N(g) \prod_{i=1}^N dH_{ii} \prod_{1 \leq i < j \leq N} d\text{Re}H_{ij} d\text{Im}H_{ij} \quad (1.3)$$

normalized in such a way that  $Z(0, 0, g; N) = 1$  for all  $g$  and  $N$ . The parameters  $p, q, g$  are *a priori* any real numbers. This model, with its logarithmic potential, can be viewed as a two-matrix generalization of the one-matrix Penner model [8]. For  $p = q$ , it has already been studied in [9] in connection with the Kazakov-Migdal model [10].

Upon setting  $g = 1/t$  and rescaling  $M \rightarrow \sqrt{t}M$ ,  $R \rightarrow \sqrt{t}R$  in the integral (1.1), we may rewrite (1.1) in the same form, but with the potential

$$W(M, R; p, q, 1/t) = p \text{Log}(1 - \sqrt{t}M) + q \text{Log}(1 - \sqrt{t}R) + MR \quad (1.4)$$

and with a different normalization of the measure, still ensuring that  $Z(0, 0, 1/t; N) = 1$ . Note that this last condition is equivalent to demanding that  $\lim_{t \rightarrow 0} Z(p, q, 1/t; N) = 1$ . If we think of  $t$  as a small parameter, we may at least formally expand the integral  $Z$  in powers of  $t$ . This expansion is expressible as a sum over Feynmann (fat)graphs, made of double-lines oriented in opposite directions and carrying a matrix index  $i \in \{1, 2, \dots, N\}$ , involving two types of vertices ( $M$ -vertices weighed by  $Np$  and  $R$ -vertices weighed by  $Nq$ , irrespectively of the valency) connected by one type of propagator ( $\langle MR \rangle$  propagator, weighed by  $t/N$ ). As usual in matrix integrals, each graph also receives a contribution  $N$  per oriented loop, from the summation over the running matrix indices. Hence we finally get the expansion

$$Z(p, q, 1/t; N) = \sum_{\text{fatgraphs } \Gamma} \frac{t^{E(\Gamma)} N^{\chi(\Gamma)} p^{V_p} q^{V_q}}{|\text{Aut}(\Gamma)|} \quad (1.5)$$

where  $E(\Gamma)$  denotes the number of edges of  $\Gamma$ ,  $\chi(\Gamma)$  its Euler characteristic,  $V_p, V_q$  its numbers of  $M$  and  $R$ -vertices respectively, and  $|\text{Aut}(\Gamma)|$  the order of its symmetry group. The sum may be restricted to connected fatgraphs by considering  $\text{Log } Z$  instead.

If we choose to color the *vertices* of the fatgraphs with two different colors, say white for the  $M$ -vertices and black for the  $R$ -vertices, the occurrence of only mixed propagators  $\langle MR \rangle$  implies that the graph is bi-colored, namely two adjacent vertices have different colors. The white (resp. black) faces receive the weight  $p$  (resp.  $q$ ). We can also consider (1.5) as an expansion over the dual fatgraphs, with now black and white alternating *faces*. So we may finally view  $\text{Log } Z$  as the partition function for connected bi-colored fatgraphs of arbitrary genus.

When compared to the class of exactly computable multi-matrix integrals, (1.1) is readily seen to be reducible to an integral over the eigenvalues of the matrices  $M$  and  $R$ , by

use of the Itzykson-Zuber integral formula [11]. However, we expect the non-polynomiality of the potential (1.2) to translate into some interesting non-locality of the model, which makes its solution slightly more subtle than that of the polynomial case.

The paper is organized as follows. The model (1.1) is solved in Sect.2, using standard orthogonal polynomial techniques. The solution involves the computation of partition functions  $Z_n$  with the same definition as (1.1), but where the integration extends over  $n \times n$  Hermitian matrices, for all values of  $n$  between 1 and  $N$ . The varying size  $n$  of the matrices introduces a new scaling variable  $z = n/N$  in the problem. Remarkably, the solution exhibits an explicit  $(p, q, z)$  symmetry relating this size scaling variable  $z$  to the coupling constants  $p$  and  $q$  of the model. This symmetry is explained by identifying our problem with that of tri-coloring the vertices of random triangulations, with the third color (say grey) receiving precisely the weight factor  $z$ . The solution is further explored by taking the large  $N$  limit, in which only planar (genus zero) diagrams contribute. The result is a non-linear partial differential equation for the free energy, involving derivatives wrt  $t$  and  $z$ . To solve it, we first derive initial data when  $z \rightarrow 0$ , in which case the free energy may be explicitly computed (the detailed derivation is done in Appendix A). In Sect.3, we present an alternative solution of our model, by first writing the integral (1.1) as a determinant, and then deriving a (discrete Hirota) bilinear equation which determines the partition function completely. In the large  $N$  limit, this takes the form of yet another non-linear partial differential equation for the free energy, involving derivatives wrt  $p$ ,  $q$ ,  $z$  and  $t$ . Combining the results of Sects.2 and 3., we are able to compute explicitly the free energy in the case  $p = q = z$ . In Sect.4, we follow a third route to express the partition function  $Z_n$  of our model as an explicit multiple sum over strictly increasing sequences of integers. We further extract the large  $N$  behavior of these sums by applying the saddle-point approximation, which yields yet another expression for the planar free energy. The latter permits to investigate the critical properties of the model. We find a first order critical point whenever  $p$ ,  $q$  and  $z$  have the same sign, with a generic singularity  $\text{Log } Z \sim (t_* - t)^{5/2}$ , characteristic of the universality class of pure two-dimensional quantum gravity. The all-genus double-scaling limit of the free energy is also derived, and shown to be governed by the Painlevé I differential equation. We conclude in Sect.5 with a discussion on the connection of our model to folding problems and with some remarks on the emergence of discrete Hirota bilinear equations in multi-matrix models.

## 2. Orthogonal Polynomial Solution

### 2.1. Preliminaries

As mentioned above, we first reduce the integral (1.1) to an integral over the eigenvalues of  $M$  and  $R$ , denoted by  $m_i$  and  $r_i$  respectively. We simply perform the change of variables  $(M; R) \rightarrow (m, U; r, V)$ , with  $M = UmU^\dagger$  and  $R = VrV^\dagger$ ,  $U, V$  two unitary matrices, and  $m = \text{diag}(m_1, \dots, m_N)$ ,  $r = \text{diag}(r_1, \dots, r_N)$ . The Jacobian of the change of variables is simply  $\Delta(m)^2 \Delta(r)^2$ , where  $\Delta(m) = \prod_{i < j} (m_i - m_j)$  denotes the Vandermonde determinant of the matrix  $m$ . The integral over the unitary matrices  $U$  and  $V$  only involves the cross-term  $-Ng\text{Tr}(MR)$  in the potential and reads

$$\int dU dV e^{-Ng\text{Tr}(MR)} = d_N(g) \frac{\det [e^{-Ngm_i r_j}]_{1 \leq i, j \leq N}}{\Delta(m) \Delta(r)} \quad (2.1)$$

by direct use of the Itzykson-Zuber integral formula. The precise value of the normalization constant  $d_N(g)$  is not important here. Substituting this into the integral (1.1), and using the antisymmetry of the Vandermonde determinant, we finally get the reduced integral

$$Z(p, q, g; N) = \int dm dr \Delta(m) \Delta(r) e^{-N\text{Tr}V(m, r; p, q, g)} \quad (2.2)$$

where the integral extends over the  $2N$  real variables  $m_i, r_i$ , and the  $N$ -dimensional measure over diagonal matrices  $l = \text{diag}(l_1, \dots, l_N)$

$$dl = \sqrt{e_N(g)} \prod_{i=1}^N dl_i \quad (2.3)$$

is normalized so as to ensure that  $Z(0, 0, g; N) = 1$ . The normalization factor  $e_N(g)$  will be computed later.

As in the standard orthogonal polynomial technique, we introduce two sets of *monic* polynomials  $p_n(x) = x^n + \text{lower degree}$ , and  $\tilde{p}_n(y) = y^n + \text{lower degree}$ , for  $n = 0, 1, 2, \dots$ , which are orthogonal with respect to the one-dimensional measure inherited from (2.2), namely

$$(p_n, \tilde{p}_m) = \int dx dy e^{-NV(x, y; p, q, g)} p_n(x) \tilde{p}_m(y) = h_n \delta_{m, n} \quad (2.4)$$

for some normalization factors  $h_n(p, q, g; N)$ , entirely fixed by the orthogonality condition. Using the multi-linearity of the determinants, we may rewrite

$$\begin{aligned} \Delta(m) &= \det [m_i^{j-1}]_{1 \leq i, j \leq N} = \det [p_{j-1}(m_i)]_{1 \leq i, j \leq N} \\ \Delta(r) &= \det [r_i^{j-1}]_{1 \leq i, j \leq N} = \det [\tilde{p}_{j-1}(r_i)]_{1 \leq i, j \leq N} \end{aligned} \quad (2.5)$$

and expand the latter. Using the orthogonality relations, the partition function is finally rewritten as

$$\boxed{Z(p, q, g; N) = N! e_N(g) \prod_{i=0}^{N-1} h_i} \quad (2.6)$$

## 2.2. $P, Q$ operators and relations

Next we write recursion relations linking the  $h_i$ 's by introducing the operators  $Q$  and  $\tilde{Q}$  of multiplication by an eigenvalue of respectively  $M$  and  $R$ , namely

$$(Qp_n)(x) = xp_n(x) \quad (\tilde{Q}\tilde{p}_n)(y) = y\tilde{p}_n(y) \quad (2.7)$$

and the operators  $P$  and  $\tilde{P}$  defined by

$$(Pp_n)(x) = (x-1)\frac{dp_n(x)}{dx} \quad (\tilde{P}\tilde{p}_n)(y) = (y-1)\frac{d\tilde{p}_n(y)}{dy} \quad (2.8)$$

Note that these differ slightly from the usual definitions for the polynomial potential case (where  $P$  and  $\tilde{P}$  are simply the derivatives wrt the corresponding eigenvalue), and that they satisfy different (non-canonical) commutation relations

$$[P, Q] = Q - 1 \quad [\tilde{P}, \tilde{Q}] = \tilde{Q} - 1 \quad (2.9)$$

Equations for the  $h_i$ 's are derived by considering various matrix elements of the operators  $P$  and  $\tilde{P}$ .

$$\begin{aligned} (Pp_n, \tilde{p}_m) &= -(p_n, \tilde{p}_m) + N((x-1)\partial_x V(x, y; p, q, g)p_n, \tilde{p}_m) \\ &= (Np-1)(p_n, \tilde{p}_m) + Ng((Q-1)p_n, \tilde{Q}\tilde{p}_m) \end{aligned} \quad (2.10)$$

where we have performed an integration by parts wrt  $x$ . Note that the particular definition of  $P$  is ad-hoc to eliminate all non-polynomial dependence on  $Q$  in the final equation. Denoting by  $A^\dagger$  the adjoint of an operator  $A$  wrt the bilinear form  $(f, Ag) = (A^\dagger f, g)$ , we can recast (2.10) into an operator identity

$$P + (1 - Np) = Ng\tilde{Q}^\dagger(Q - 1) \quad (2.11)$$

Analogously, we have

$$\tilde{P} + (1 - Nq) = NgQ^\dagger(\tilde{Q} - 1) \quad (2.12)$$

The operator relations (2.9), (2.11) and (2.12) determine the  $h_i$ 's entirely.

Note that in our definitions,  $P, Q$  and  $\tilde{P}, \tilde{Q}$  act on two different bases  $(p_n)$  and  $(\tilde{p}_n)$  of the space of polynomials of one variable. Let us now write the actions of the operators  $P, Q, \tilde{P}^\dagger$  and  $\tilde{Q}^\dagger$  on the *same* basis  $(p_n)$ . We have

$$\begin{aligned}
(Qp_n)(x) &= \sum_{k \geq -1} Q_{n,k} p_{n-k}(x) \\
(Pp_n)(x) &= \sum_{k \geq 0} P_{n,k} p_{n-k}(x) \\
(\tilde{Q}^\dagger p_n)(x) &= \sum_{k \geq -1} \tilde{Q}_{n,k} p_{n+k}(x) \\
(\tilde{P}^\dagger p_n)(x) &= \sum_{k \geq 0} \tilde{P}_{n,k} p_{n+k}(x)
\end{aligned} \tag{2.13}$$

where the first terms are given by

$$\begin{aligned}
Q_{n,-1} &= 1 \\
P_{n,0} &= n \\
\tilde{Q}_{n,-1} &= \frac{h_n}{h_{n-1}} \\
\tilde{P}_{n,0} &= n
\end{aligned} \tag{2.14}$$

The first two lines of (2.14) follow directly from the definitions (2.8), whereas the last two lines are easily derived from the identities

$$\begin{aligned}
(\tilde{Q}^\dagger p_n, \tilde{p}_{n-1}) &= (p_n, \tilde{Q} \tilde{p}_{n-1}) = h_n = \tilde{Q}_{n,-1} h_{n-1} \\
(\tilde{P}^\dagger p_n, \tilde{p}_n) &= (p_n, \tilde{P} \tilde{p}_n) = n h_n = \tilde{P}_{n,0} h_n
\end{aligned} \tag{2.15}$$

We obtain all the relations between the coefficients in (2.13), by expressing (2.9) and (2.11)(2.12) as

$$\begin{aligned}
[P, Q] &= Q - 1 \\
[\tilde{Q}^\dagger, \tilde{P}^\dagger] &= \tilde{Q}^\dagger - 1 \\
P + (1 - Np) &= Ng \tilde{Q}^\dagger (Q - 1) \\
\tilde{P}^\dagger + (1 - Nq) &= Ng (\tilde{Q}^\dagger - 1) Q
\end{aligned}$$

(2.16)

and letting these operators act on  $(p_n)$ , namely

$$\begin{aligned} Q_{n,k} - \delta_{k,0} &= \sum_{m \geq -1} Q_{n,m} P_{n-m,k-m} - P_{n,k-m} Q_{n-k+m,m} \\ \tilde{Q}_{n,k} - \delta_{k,0} &= \sum_{m \geq -1} \tilde{P}_{n,k-m} \tilde{Q}_{n+k-m,m} - \tilde{Q}_{n,m} \tilde{P}_{n+m,k-m} \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} (1 - Np)\delta_{k,0} + P_{n,k} + Ng\tilde{Q}_{n,-k} &= Ng \sum_{m \geq -1} \tilde{Q}_{n-m,m-k} Q_{n,m} \\ (1 - Nq)\delta_{k,0} + \tilde{P}_{n,-k} + NgQ_{n,k} &= Ng \sum_{m \geq -1} \tilde{Q}_{n-m,m-k} Q_{n,m} \end{aligned} \quad (2.18)$$

This gives an infinite set of equations for the coefficients of (2.13). A drastic simplification of (2.18) follows from subtracting the third from the fourth line of (2.16), cancelling the crossed term  $\tilde{Q}^\dagger Q$ . This results in the operator identity

$$A = P - NgQ - Np = (\tilde{P} - Ng\tilde{Q} - Nq)^\dagger = \tilde{A}^\dagger \quad (2.19)$$

where we have formed two convenient linear combinations  $A$  and  $\tilde{A}$  of the  $P$ ,  $Q$  and  $\tilde{P}$ ,  $\tilde{Q}$  respectively. When expressed on the basis  $(p_n)$ , (2.19) yields

$$P_{n,k} - NgQ_{n,k} - Np\delta_{k,0} = \tilde{P}_{n,-k} - Ng\tilde{Q}_{n,-k} - Nq\delta_{k,0} \quad (2.20)$$

Taking  $k > 1$  (resp.  $k < -1$ ) in this formula, in which case the rhs (resp. the lhs) vanishes, this gives

$$P_{n,k} = NgQ_{n,k} \quad \tilde{P}_{n,k} = Ng\tilde{Q}_{n,k} \quad (2.21)$$

for all  $k > 1$ . Taking now  $k = 1$  and  $k = -1$  in (2.20) leads to

$$\begin{aligned} P_{n,1} - NgQ_{n,1} &= -Ng\tilde{Q}_{n,-1} \\ \tilde{P}_{n,1} - Ng\tilde{Q}_{n,1} &= -NgQ_{n,-1} \end{aligned} \quad (2.22)$$

while  $k = 0$ , (2.20) gives

$$Q_{n,0} - \tilde{Q}_{n,0} = \frac{q - p}{g} \quad (2.23)$$

But even after eliminating  $P_{n,k}$ ,  $\tilde{Q}_{n,k}$  and  $\tilde{P}_{n,k}$ , (2.17)(2.18) remain an infinite set of quadratic equations for the  $Q_{n,k}$ .



In the next section, we shall derive a set of equations involving only the first values of the index  $k$ , at the cost of taking derivatives wrt  $g$ . We shall make use only of Eqs. (2.14) and (2.23), together with the first line of (2.17) for  $k = 0$ , which yields

$$Q_{n,0} - 1 = P_{n+1,1} - P_{n,1} \quad (2.24)$$

This equation enables us to solve for  $P_{n,1}$  in terms of the  $Q_{i,0}$ , given that  $P_{0,1} = 0$  by definition. Introducing the quantity

$$v_n = \sum_{k=0}^{n-1} Q_{k,0} \quad (2.25)$$

we find that

$$P_{n,1} = v_n - n \quad (2.26)$$

### 2.3. The main recursion relation

In this section, we take advantage of the simple dependence of  $Z$  on  $g$  to derive differential equations for the first coefficients  $Q_{n,k}$ , which, supplemented to the results of the previous section, yield a recursion relation for  $Z$ .

Let us evaluate the following derivatives wrt  $g$  (the notation  $\partial_g$  stands for  $\partial/\partial g$ )

$$\begin{aligned} \partial_g(p_n, \tilde{p}_{n-1}) &= 0 = (\partial_g p_n, \tilde{p}_{n-1}) + (p_n, \partial_g \tilde{p}_{n-1}) - N(Qp_n, \tilde{Q}p_{n-1}) \\ &= (\partial_g p_n, \tilde{p}_{n-1}) - N(\tilde{Q}^\dagger Qp_n, \tilde{p}_{n-1}) \\ \partial_g(p_n, \tilde{p}_n) &= \partial_g h_n = -N(\tilde{Q}^\dagger Qp_n, \tilde{p}_n) \end{aligned} \quad (2.27)$$

where we have used the fact that  $\partial_g p_n$  has degree  $\leq n-1$ , and the orthogonality of the  $p$ 's and  $\tilde{p}$ 's. To compute  $(\partial_g p_n, \tilde{p}_{n-1})$ , let us write  $p_n(x) = x^n - \lambda_n x^{n-1} + \dots$ . Using the definition of  $Q$  (2.7), we easily see that  $\lambda_n = v_n$ , given by (2.25). Indeed, we write

$$\begin{aligned} p_{n+1}(x) &= x^{n+1} - \lambda_{n+1} x^n + O(x^{n-1}) \\ &= Qp_n(x) - Q_{n,0} x^n + O(x^{n-1}) \\ &= x^{n+1} - (\lambda_n + Q_{n,0}) x^n + O(x^{n-1}) \end{aligned} \quad (2.28)$$

and the result follows. Hence

$$(\partial_g p_n, \tilde{p}_{n-1}) = -h_{n-1} \partial_g v_n \quad (2.29)$$

Finally, let us express  $N(\tilde{Q}^\dagger Q p_n, \tilde{p}_m)$  by using (2.11), namely that  $N\tilde{Q}^\dagger Q = N\tilde{Q}^\dagger + (P + 1 - Np)/g$ , hence

$$\begin{aligned} N(\tilde{Q}^\dagger Q p_n, \tilde{p}_{n-1}) &= h_{n-1} \left( N\tilde{Q}_{n,-1} + \frac{1}{g} P_{n,1} \right) \\ N(\tilde{Q}^\dagger Q p_n, \tilde{p}_n) &= h_n \left( N\tilde{Q}_{n,0} + \frac{1}{g} (P_{n,0} + 1 - Np) \right) \end{aligned} \quad (2.30)$$

and finally, using the explicit values (2.14) and (2.26), we get the differential equations

$$\begin{aligned} (\partial_g + \frac{1}{g})v_n &= \frac{n}{g} - N \frac{h_n}{h_{n-1}} \\ \partial_g \text{Log } h_n &= \frac{1}{g} (Np - n - 1) - N\tilde{Q}_{n,0} \\ &= \frac{1}{g} (Nq - n - 1) - NQ_{n,0} \end{aligned} \quad (2.31)$$

where we have used (2.23) to get the last line. Let us introduce

$$F_n = \sum_{k=0}^{n-1} \text{Log } h_k \quad (2.32)$$

so that

$$\text{Log } Z(p, q, g; N) = F_N + \text{Log}(e_N(g)N!) \quad (2.33)$$

then after summation, the second equation of (2.31) becomes

$$\partial_g F_n = \frac{1}{2g} n(2Nq - n - 1) - Nv_n \quad (2.34)$$

After elimination of  $v_n$  between (2.31) and (2.33), we are left with

$$\frac{1}{N^2} (\partial_g + \frac{1}{g}) \partial_g F_n = \frac{h_n}{h_{n-1}} - \frac{n}{Ng} \quad (2.35)$$

In terms of the variable  $t = 1/g$ , and upon the definition

$$\begin{aligned} \alpha_n &= \frac{h_n}{h_{n-1}} \quad n = 1, 2, \dots \\ \alpha_0 &= h_0 \end{aligned} \quad (2.36)$$

we finally have the system of equations

$$\begin{aligned} F_n &= \sum_{k=0}^{n-1} (n-k) \text{Log } \alpha_k \\ \alpha_n &= n \frac{t}{N} + \frac{t^2}{N^2} (t \partial_t)^2 F_n \end{aligned}$$

(2.37)

Given the initial data  $\alpha_0 = h_0(p, q, 1/t; N)$ , this set of equation gives a recursion relation which determines  $F_n$  for all  $n$ , and in particular  $F_N$ .

The normalization factor  $e_N(g)$  in (2.33) still has to be fixed. It is obtained from the solution  $(\alpha_n^{(0)}, F_n^{(0)})$  of the system (2.37) for the case  $p = q = 0$ . In that case, the initial data reads

$$\alpha_0^{(0)} = h_0(0, 0, 1/t; N) = \int e^{-Nxy/t} dx dy = \frac{t}{N} \quad (2.38)$$

We then have

$$\alpha_n^{(0)} = n \frac{t}{N} \quad (2.39)$$

for  $n \geq 1$ , and

$$\begin{aligned} F_n^{(0)} &= \text{Log} \frac{t}{N} + \sum_{k=1}^{n-1} (n-k) \text{Log} \left( \frac{kt}{N} \right) \\ &= \text{Log} \left( \left( \frac{t}{N} \right)^{n(n+1)/2} \prod_{j=0}^{n-1} j! \right) \end{aligned} \quad (2.40)$$

Now the normalization  $e_N(g)$  in (2.33) ensures that  $Z(0, 0, 1/t; N) = 1$ , hence we have

$$\text{Log} Z(p, q, 1/t; N) = F_N - F_N^{(0)} \quad (2.41)$$

Upon comparison with (2.33), we deduce that

$$e_N(1/t) = \frac{1}{(t/N)^{N(N+1)/2} \prod_{j=0}^N j!} \quad (2.42)$$

Introducing the renormalized quantities

$$\begin{aligned} a_n &= \frac{N\alpha_n}{nt} \quad \text{for } n = 1, 2, \dots \\ a_0 &= \frac{N\alpha_0}{t} \end{aligned} \quad (2.43)$$

and the free energy

$$f_n = F_n - F_n^{(0)} \quad (2.44)$$

such that

$$f_N = \text{Log} Z(p, q, 1/t; N) \quad (2.45)$$

we finally have the recursive system, for  $n \geq 1$

$$\boxed{\begin{aligned} f_n &= \sum_{k=0}^{n-1} (n-k) \text{Log } a_k \\ a_n &= 1 + \frac{t}{N} (t\partial_t)^2 \frac{f_n}{n} \end{aligned}} \quad (2.46)$$

with the initial data

$$\boxed{\begin{aligned} a_0 &= \frac{N}{t} \int dx dy (1-x)^{-Np} (1-y)^{-Nq} e^{-Nxy/t} \\ &= \sum_{k \geq 0} \frac{1}{k!} \frac{\Gamma(Np+k)\Gamma(Nq+k)}{\Gamma(Np)\Gamma(Nq)} \left(\frac{t}{N}\right)^k \end{aligned}} \quad (2.47)$$

Note that if  $-Np$  and/or  $-Nq$  is a positive integer, the above sum truncates.

At this stage, it is interesting to note that the quantity  $f_n$  which appears in the recursion relation (2.46) has the following simple interpretation. Indeed, for  $n \geq 1$ , we have

$$f_n = \text{Log } Z_n \quad (2.48)$$

where  $Z_n = n! e_n(g) \prod_{i=0}^{n-1} h_i$  is the integral (1.1) taken over  $n \times n$  matrices, but with the *same* potential (1.2), with  $N$  as a prefactor. The quantity  $f_n$  is thus the corresponding free energy. The normalization factor  $e_n(g)$ , which ensures that  $Z_n = 1$  for  $p = q = 0$  is a straightforward extension of (2.42)

$$e_n(g) = \frac{(Ng)^{n(n+1)/2}}{\prod_{j=0}^n j!} \quad (2.49)$$

In terms of fatgraphs, we now have a weight  $n$  instead of  $N$  per oriented loop, leading to

$$Z_n(p, q, 1/t; N) = \sum_{\text{fatgraphs } \Gamma} \frac{\left(\frac{n}{N}\right)^{F(\Gamma)} t^{E(\Gamma)} N^{\chi(\Gamma)} p^{V_p} q^{V_q}}{|\text{Aut}(\Gamma)|} \quad (2.50)$$

where  $F(\Gamma)$  denotes the number of faces of the graph. Therefore we may think of the new scaling variable  $z = n/N$  as an activity per *face* of the bi-colored graph.

#### 2.4. Solution of the main recursion relation

To solve (2.46), we first eliminate  $a_n$  by noting that

$$a_n = \frac{Z_{n+1}Z_{n-1}}{Z_n^2} \quad (2.51)$$

for all  $n \geq 0$  by setting  $Z_{-1} = Z_0 = 1$ . This is obtained by inverting the relations  $Z_n = \prod_{k=0}^{n-1} a_k^{n-k}$ . In terms of  $Z_n$ , the recursion relation (2.46) reads

$$\boxed{\frac{Z_{n-1}Z_{n+1}}{Z_n^2} - 1 = \frac{t}{nN}(t\partial_t)^2 \text{Log } Z_n} \quad (2.52)$$

for all  $n \geq 1$ . The initial data are  $Z_{-1} = Z_0 = 1$ , and  $Z_1 = a_0$  as in (2.47).

Let us first solve (2.46) in the case

$$a = Np = -1 \quad (2.53)$$

and  $b = Nq$  arbitrary. The initial data (2.47) reads

$$a_0 = 1 - b\frac{t}{N} \quad (2.54)$$

The solution of (2.52) takes the form

$$Z_n = 1 + \sum_{k=1}^n \binom{n}{k} b(b-1)\dots(b-k+1) \left(-\frac{t}{N}\right)^k \quad (2.55)$$

for  $n \geq 0$  and  $Z_{-1} = 1$ , as easily checked by explicitly differentiating the rhs<sup>1</sup>. In particular, the solution for  $n = N$  reads

$$Z(-1/N, q, 1/t; N) = 1 + \sum_{k=1}^N \binom{N}{k} (Nq)(Nq-1)\dots(Nq-k+1) \left(-\frac{t}{N}\right)^k \quad (2.56)$$

A similar reasoning permits to solve the case

$$a = Np = 1 \quad (2.57)$$

---

<sup>1</sup> This result will also be proved directly in Sect.5.2 below.

and  $b = Nq$  arbitrary, with now

$$Z_n = 1 + \sum_{k=1}^{\infty} \binom{n+k-1}{k} b(b+1)\dots(b+k-1) \left(\frac{t}{N}\right)^k \quad (2.58)$$

also satisfying (2.52). The corresponding solution for  $n = N$  reads

$$Z(1/N, q, 1/t; N) = 1 + \sum_{k=1}^{\infty} \binom{N+k-1}{k} Nq(Nq+1)\dots(Nq+k-1) \left(\frac{t}{N}\right)^k \quad (2.59)$$

More generally, we can use the recursion (2.52) to expand  $f_n = \text{Log } Z_n(p, q, 1/t; N)$  as a formal power series of  $t$

$$f_n(p, q, 1/t; N) = \sum_{k \geq 1} \omega_{n,k}(a, b) \left(\frac{t}{N}\right)^k \quad (2.60)$$

with  $a = Np$ ,  $b = Nq$ . The initial conditions  $Z_0 = 1$  and  $Z_1 = a_0$  translate into the conditions

$$\begin{aligned} \omega_{0,k}(a, b) &= 0 \\ \sum_{k \geq 1} \omega_{1,k}(t/N)^k &= \text{Log } a_0 \end{aligned}$$

(2.61)

with  $a_0$  as in (2.47). The recursion relation (2.52) turns into a recursion relation for the  $\omega$ 's

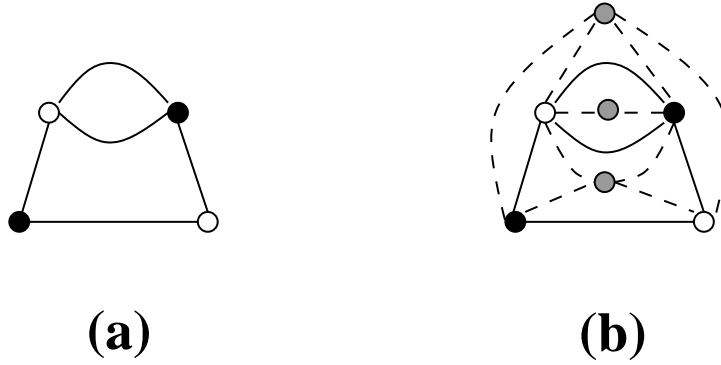
$$\begin{aligned} \omega_{n+1,k} &= 2\omega_{n,k} - \omega_{n-1,k} \\ &+ \sum_{r \geq 1} \frac{(-1)^{r-1}}{r n^r} \sum_{\substack{k_1, \dots, k_r \geq 1 \\ \sum k_i = k-r}} (k_1 \dots k_r)^2 \omega_{n,k_1} \dots \omega_{n,k_r} \end{aligned}$$

(2.62)

Together with the initial conditions (2.61), this determines the coefficients  $\omega_{n,k}(a, b)$  completely. It is easy to show that these are polynomials of  $a$ ,  $b$ , and  $n$  for each value of  $k$ .

The first few of them read

$$\begin{aligned}
\omega_{n,1} &= nab \\
\omega_{n,2} &= \frac{nab}{2}(n + a + b) \\
\omega_{n,3} &= \frac{nab}{3}(n^2 + 3(a + b)n + a^2 + 3ab + b^2 + 1) \\
\omega_{n,4} &= \frac{nab}{4}(n^3 + 6(a + b)n^2 + (6a^2 + 17ab + 6b^2 + 5)n \\
&\quad + (a + b)(a^2 + 5ab + b^2 + 5)) \\
\omega_{n,5} &= \frac{nab}{5}(n^4 + 10(a + b)n^3 + 5(4a^2 + 11ab + 4b^2 + 3)n^2 \\
&\quad + 5(a + b)(2a^2 + 9ab + 2b^2 + 8)n \\
&\quad + a^4 + 10a^3b + 20a^2b^2 + 10ab^3 + b^4 + 15a^2 + 40ab + 15b^2 + 8)
\end{aligned} \tag{2.63}$$



**Fig. 1:** The construction of the tri-colored graph  $\tilde{\Gamma}$  (b) corresponding to the bi-colored graph  $\Gamma$  (a). Each face is subdivided into triangles, by adding a new central vertex, colored in grey. The face-weight  $z$  is re-affected to each such new grey vertex.

Remarkably, the result (2.63) reveals that each coefficient  $\omega_{n,k}(a, b)$  is *totally symmetric* in the three variables  $(a, b, n)$ , or equivalently the three variables  $(p, q, z = n/N)$ . Let us prove this property directly on  $f_n(p, q, 1/t; N)$  (2.60) by considering its fatgraph expansion (2.50) (restricted to connected graphs). For each fatgraph  $\Gamma$  with black and white vertices in the expansion (2.50), let us construct one grey vertex in the middle of each face (see Fig.1), and connect it to each (black or white) vertex around the face. As black and white vertices alternate around each face, this creates an even number of edges. Each grey vertex receives a contribution  $z$ , whereas white vertices are weighed by  $p$  and black ones by  $q$ . We end up with a fatgraph  $\tilde{\Gamma}$  whose vertices are tri-colored (in white, black and grey), and

whose faces are all triangles with a vertex of each color. The automorphism group remains unchanged ( $|\text{Aut}(\tilde{\Gamma})| = |\text{Aut}(\Gamma)|$ ), so as the Euler characteristic ( $\chi(\tilde{\Gamma}) = \chi(\Gamma)$ ), hence we get the expansion

$$f_n(p, q, 1/t; N) = \sum_{\substack{\text{conn. tricol.} \\ \text{fatgraphs } \tilde{\Gamma}}} t^{\frac{F}{2}} \frac{p^{V_w} q^{V_b} z^{V_g} N^\chi}{|\text{Aut}(\tilde{\Gamma})|} \quad (2.64)$$

where  $F$  denotes the total number of (triangular) faces of  $\tilde{\Gamma}$  ( $F = 2E(\Gamma)$ ), and  $V_w, V_b, V_g$  respectively its numbers of white, black and grey vertices. The form (2.64) is explicitly symmetric in the three variables  $(p, q, z)$ , that is in the three variables  $(a, b, n)$ .

The  $a \leftrightarrow b$  symmetry was clear from the integral (1.1). Another direct proof of the symmetry  $n \leftrightarrow a$  will be given in Sect.4.2 below. As an additional check, we may compare  $Z_1(a = n, b) = a_0(a = n, b, t)$  as given by (2.47), to  $Z_n(1, b)$  found in (2.58): the two sums are indeed identical.

## 2.5. Large $N$ limit

In the large  $N$  limit, the free energy  $\text{Log } Z$  is dominated by the planar (genus zero) fatgraphs, hence a leading contribution

$$\text{Log } Z(p, q, 1/t; N) \sim N^2 F_0(p, q, t) \quad (2.65)$$

when  $N \rightarrow \infty$ . Introducing the rescaled variable  $z = n/N$ , we may assume that  $a_n$  tend to a smooth function  $a(z; p, q, t)$  of  $z \in [0, 1]$  in the large  $N$  limit. This is in agreement with the assumption that  $f_n/N^2$  also tends to a smooth function  $f(z; p, q, t)$ . Indeed, (2.46) then implies that

$$a(z; p, q, t) = 1 + \frac{t}{z} (t\partial_t)^2 f(z; p, q, t) \quad (2.66)$$

According to (2.50), the continuous quantity  $z$  is nothing but a fugacity associated with the faces of the (planar) fatgraphs.

The recursion relation (2.52) reads, for  $\epsilon = 1/N$  and  $f(z) \equiv f(z; p, q, t)$

$$e^{N^2(f(z+\epsilon)+f(z-\epsilon)-2f(z))+O(1)} - 1 = \frac{t}{nN} (t\partial_t)^2 (N^2 f(z) + O(1)) \quad (2.67)$$

hence in the limit  $N \rightarrow \infty$

$$\partial_z^2 f = \text{Log}(1 + \frac{t}{z} (t\partial_t)^2 f)$$

(2.68)



The partial differential equation (2.68) however does not determine  $f(z; p, q, t)$  completely. We need to supplement it by some initial condition. This is easily done by use of the formal small  $t$  series expansion of  $f$ . From the large  $N$  limit of (2.60)(2.63) (with  $a = Np$ ,  $b = Nq$ ), we indeed find that

$$\begin{aligned} f_n &= nab \frac{t}{N} + O(t^2) \\ &= N^2 pqz t + O(t^2) \end{aligned} \quad (2.69)$$

hence we get the initial condition

$$f(z; p, q, t) = pqz t + O(t^2) \quad (2.70)$$

This can be used to feed the differential equation (2.68). Indeed, writing

$$f(z; p, q, t) = \sum_{k \geq 1} \omega_k(z; p, q) t^k \quad (2.71)$$

with  $\omega_1 = pqz$ , the equation (2.68) turns into a recursion relation

$$\partial_z^2 \omega_k = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r z^r} \sum_{\substack{k_1, \dots, k_r \geq 1 \\ \sum k_i = k-r}} k_1^2 \dots k_r^2 \omega_{k_1} \dots \omega_{k_r} \quad (2.72)$$

for  $k \geq 2$ . This is nothing but the continuum limit of the recursion relation (2.62). Eq.(2.72) however involves two integrations wrt  $z$  at each step, hence the introduction of many integration constants. To fix them, we need to know the values of  $f(0; p, q, t)$  and  $\partial_z f(0; p, q, t)$ . The latter are readily obtained by analyzing the large  $N$  limit of the initial condition (2.47), namely we write

$$\begin{aligned} Z_1 &= a_0 = \sum_{k \geq 0} \frac{1}{k!} \frac{\Gamma(Np+k)\Gamma(Nq+k)}{\Gamma(Np)\Gamma(Nq)} \left( \frac{t}{N} \right)^k \\ &\sim N \int_0^\infty d\alpha e^{N S(\alpha; p, q, t)} \end{aligned} \quad (2.73)$$

where we have replaced the summation over  $k = \alpha N$  by an integral over the positive real numbers  $\alpha$ , and used the Stirling formula to derive the effective action

$$\begin{aligned} S(\alpha; p, q, t) &= (p + \alpha)(\text{Log}(p + \alpha) - 1) + (q + \alpha)(\text{Log}(q + \alpha) - 1) + \alpha \text{Log } t \\ &\quad - \alpha(\text{Log } \alpha - 1) - p(\text{Log } p - 1) - q(\text{Log } q - 1) \end{aligned} \quad (2.74)$$

The integral is dominated by the saddle-point  $dS/d\alpha(\alpha^*) = 0$ , namely

$$t(\alpha^* + p)(\alpha^* + q) = \alpha^* \quad (2.75)$$

We only retain the solution leading to a maximum of  $S$ , namely

$$\alpha^* = \frac{1 - t(p + q)}{2t} - \frac{1}{2t} \sqrt{(1 - t(p + q))^2 - 4pqt^2} \quad (2.76)$$

Finally, the free energy  $f_1 = \text{Log } Z_1$  behaves for large  $N$  as

$$\begin{aligned} f_1(p, q, t) &\sim N S(\alpha^*) \\ &= N \left( p \text{Log}\left(1 + \frac{\alpha^*}{p}\right) + q \text{Log}\left(1 + \frac{\alpha^*}{q}\right) - \alpha^* \right) \\ &\equiv N \varphi_1(p, q, t) \end{aligned} \quad (2.77)$$

Let us now interpret the result (2.77). As  $n = 1$ , we have  $z = 1/N$ . The fact that  $f_1$  is of order  $N = N^2 z$  shows that

$$f(z; p, q, t) = z\varphi_1(p, q, t) + O(z^2) \quad (2.78)$$

for small  $z$ , hence that

$$\begin{aligned} f(0; p, q, t) &= 0 \\ \partial_z f(0; p, q, t) &= \varphi_1(p, q, t) = \sum_{k \geq 1} \mu_k(p, q) t^k \end{aligned} \quad (2.79)$$

with  $\varphi_1$  as in (2.76)(2.77). The coefficients  $\mu_k(p, q)$  are computed in Appendix A, and read

$$\mu_k(p, q) = \frac{1}{k^2} \sum_{j=1}^k \binom{k}{j} \binom{k}{j-1} p^j q^{k+1-j} \quad (2.80)$$

The quantity  $\varphi_1(p, q, t)$  can be viewed as the free energy associated with bi-colored graphs with exactly one vertex, i.e. bi-colored trees. As clear from (2.76), it develops a singularity at a critical value for  $t$  equal to

$$t_*(p, q, 0) = \frac{1}{(\sqrt{p} + \sqrt{q})^2} \quad (2.81)$$

For  $t \rightarrow t_*(p, q, 0)$ , we find (see Appendix A)

$$\varphi_1(p, q, t) \sim (t_* - t)^{3/2} \quad (2.82)$$

As we shall see however, this behavior for the free energy, valid only in the limit  $z \rightarrow 0$ , is not generic.

With the initial data (2.79)(2.80), we may now write the differential equation (2.72) in integrated form with  $\omega_k(z) \equiv \omega_k(z; p, q)$

$$\omega_k(z) = z \mu_k(p, q) + \int_0^z dx \int_0^x dy \sum_{r \geq 1} \frac{(-1)^{r-1}}{r y^r} \sum_{\substack{k_1, \dots, k_r \geq 1 \\ \sum k_i = k-r}} k_1^2 \dots k_r^2 \omega_{k_1}(y) \dots \omega_{k_r}(y) \quad (2.83)$$

This is now a true recursion relation on  $k$ , which determines all  $\omega$ 's. With the initial condition  $\omega_0(z; p, q) = 0$ , we find

$$\begin{aligned} \omega_1 &= pqz \\ \omega_2 &= \frac{pqz}{2}(z + p + q) \\ \omega_3 &= \frac{pqz}{3}(z^2 + 3(p+q)z + p^2 + 3pq + q^2) \\ \omega_4 &= \frac{pqz}{4}(z^3 + 6(p+q)z^2 + (6p^2 + 17pq + 6q^2)z + (p+q)(p^2 + 5pq + q^2)) \\ \omega_5 &= \frac{pqz}{5}(z^4 + 10(p+q)z^3 + (20p^2 + 55pq + 20q^2)z^2 \\ &\quad + 5(p+q)(2p^2 + 9pq + 2q^2)z + p^4 + 10p^3q + 20p^2q^2 + 10pq^3 + q^4) \\ \omega_6 &= \frac{pqz}{6}(z^5 + 15(p+q)z^4 + 5(10p^2 + 27pq + 10q^2)z^3 \\ &\quad + 2(p+q)(25p^2 + 106pq + 25q^2)z^2 + (15p^4 + 135p^3q + 262p^2q^2 + 135pq^3 + 15q^4)z \\ &\quad + (p+q)(p^4 + 14p^3q + 36p^2q^2 + 14pq^3 + q^4)) \end{aligned} \quad (2.84)$$

For  $z = 1$ , this gives the first few terms in the expansion of the planar free energy (2.65) as a formal power series of  $t$ . In this expansion, the coefficient of  $t^k p^r q^s$  corresponds to the planar connected (dual) black and white fatgraphs with  $r$  white faces,  $s$  black faces, and  $k$  edges separating them, weighed by the inverse of the order of their symmetry group. For  $z$  arbitrary, the coefficient of  $z^m$  singles out those black and white graphs with exactly  $m$  vertices. Since we are dealing with planar graphs, one has  $m + r + s = k + 2$ . This explains why  $\omega_k(z; p, q)$  is a *homogeneous* polynomial of  $p, q, z$ , of degree  $k + 2$ . Note also that (2.84) agrees with the large  $N$  limit of (2.63), i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \omega_{zN, k}(Np, Nq) = \omega_k(z; p, q) \quad (2.85)$$

Finally, the result (2.84) is explicitly symmetric in the three variables  $(p, q, z)$ , in agreement with the expected symmetry

$$f(z; p, q, t) = f(z; q, p, t) = f(p; z, q, t) \quad (2.86)$$

### 3. Determinant form and discrete Hirota equation

#### 3.1. The partition function as a determinant

Throughout this section, we use the shorthand notation  $Z_n(a, b)$  for the partition function  $Z_n(p, q, g; N)$  (1.1) for an integral over  $n \times n$  matrices, and with  $a = Np$ ,  $b = Nq$ .

We start from the reduced integral (2.2)(2.3), extending over the  $2n$  eigenvalues  $m_i$  and  $r_i$

$$Z_n(a, b) = e_n(g) \int \prod_{i=1}^n dm_i dr_i (1 - m_i)^{-a} (1 - r_i)^{-b} \Delta(m) \Delta(r) e^{-Ng \sum_{i=1}^n m_i r_i} \quad (3.1)$$

From the definition of the Vandermonde determinant, we have  $\Delta(1-m) = (-1)^{n(n-1)/2} \Delta(m)$ , hence we may replace  $m \rightarrow 1-m$  and  $r \rightarrow 1-r$  in both Vandermonde determinants without altering (3.1). Moreover,

$$\begin{aligned} \prod_{i=1}^n (1 - m_i)^{-a} \Delta(1 - m) &= \det \left[ (1 - m_i)^{j-a-1} \right]_{1 \leq i, j \leq n} \\ &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n (1 - m_i)^{\sigma(i)-a-1} \end{aligned} \quad (3.2)$$

and we have an analogous formula for  $r$  (with  $b$  instead of  $a$ ). Substituting these expansions into (3.1), we get

$$\begin{aligned} Z_n(a, b) &= e_n(g) \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma\tau) \\ &\times \prod_{i=1}^n \int dm_i dr_i (1 - m_i)^{\sigma(i)-a-1} (1 - r_i)^{\tau(i)-b-1} e^{-Ng m_i r_i} \\ &= n! e_n(g) \sum_{\nu \in S_n} \text{sgn}(\nu) \prod_{i=1}^n \int dx dy (1 - x)^{i-a-1} (1 - y)^{\nu(i)-b-1} e^{-Ng xy} \end{aligned} \quad (3.3)$$

where we have set  $\nu = \tau\sigma^{-1}$ , with the same signature as  $\sigma\tau$ , and explicitly factored out the sum over  $\sigma$ . Moreover, the dummy integration variables have been rebaptized  $x$  and  $y$ . The partition function takes therefore the form

$$Z_n(a, b) = n! e_n(g) \phi_n(a, b) \quad (3.4)$$

where  $\phi_n(a, b)$  is the  $n \times n$  determinant

$$\phi_n(a, b) = \det \left[ \int dx dy (1 - x)^{i-a-1} (1 - y)^{j-b-1} e^{-Ng xy} \right]_{1 \leq i, j \leq n}$$

(3.5)

### 3.2. Discrete Hirota equation

The technique used in this section to derive a discrete Hirota equation for  $\phi_n$  is borrowed from [12]. The argument is based on a general quadratic relation satisfied by the minors of the determinant  $D$  of any  $(n+1) \times (n+1)$  matrix. If we denote by  $D_{i,j}$  the  $n \times n$  minor of  $D$  obtained by erasing the  $i$ -th row and  $j$ -th column, and  $D_{i_1, i_2; j_1, j_2}$  the  $(n-1) \times (n-1)$  minor obtained by removing the rows  $i_1, i_2$  and columns  $j_1, j_2$ , we have the relation

$$D D_{1, n+1; 1, n+1} = D_{n+1, n+1} D_{1, 1} - D_{1, n+1} D_{n+1, 1} \quad (3.6)$$

This is proved for instance by collecting the monomials in the expansions of the various minors. This may also be viewed as a particular case of the Plücker relations [13]. When applied to  $\phi_{n+1}(a+1, b+1)$ , (3.6) immediately translates into the quadratic relation

$$\phi_{n+1}(a+1, b+1)\phi_{n-1}(a, b) = \phi_n(a+1, b+1)\phi_n(a, b) - \phi_n(a, b+1)\phi_n(a+1, b) \quad (3.7)$$

where we have absorbed the various restrictions on the matrix indices  $i, j$  into corresponding shifts of  $a$  and  $b$ , while the index of  $\phi$  refers to the size of the minor. The equation (3.7), is known as a discrete Hirota equation, playing a central role in integrable systems (see [13] for the study of analogous equations).

We may recast (3.7) in terms of  $Z_n$ , using (3.5) and the value (2.49) of  $e_n(g)$  satisfying

$$\frac{(n+1)!e_{n+1}(g)(n-1)!e_{n-1}(g)}{(n!e_n(g))^2} = \frac{1}{Ng} \frac{n!}{(n-1)!} = \frac{n}{Ng} \quad (3.8)$$

This leads to

$$n \frac{t}{N} Z_{n+1}(a+1, b+1) Z_{n-1}(a, b) = Z_n(a+1, b+1) Z_n(a, b) - Z_n(a, b+1) Z_n(a+1, b)$$

(3.9)

The equation (3.9) gives a completely different information on  $Z_n(a, b)$  or  $f_n(a, b)$  than the recursion relation (2.52). Indeed, the latter determines the dependence of  $Z_n$  on  $a, b$  from the initial data  $Z_1$  and through a recursive process, the former instead gives a three-dimensional recursion relation on the parameters  $a, b$  and  $n$ . In terms of the free energy

$$f_n(a, b) = \text{Log } Z_n(a, b) \quad (3.10)$$

and using the difference operator

$$\delta_x f(x) = f(x+1) - f(x) \quad (3.11)$$

we can rewrite (3.9) as

$$\delta_a \delta_b f_n(a, b) = -\text{Log}\left(1 - n \frac{t}{N} e^{\delta_n f_n(a+1, b+1) - \delta_n f_{n-1}(a, b)}\right)$$

(3.12)

Substituting for  $f_n(a, b)$  the small  $t$  expansion (2.60), this gives yet another recursion relation for the coefficients  $\omega_{n,k}(a, b)$ , with the form

$$\delta_a \delta_b \omega_{n,k+1} = \varphi(\omega_{j,l} | j \in \{n+1, n, n-1\}; l \leq k) \quad (3.13)$$

This is a recursion over  $k$ , with the initial conditions  $\omega_{n,0} = 0$  and  $\omega_{n,1} = nab$ . However, it involves a discrete integration wrt  $a$  and  $b$ , namely we have to solve an equation of the form  $\delta_a \delta_b l = h$  at each step. As  $h$  is a polynomial of  $a$  and  $b$ , this is readily done by noticing that the polynomial  $\omega_{n,k}(a, b)$  is always a factor of  $ab$ . Indeed, from the original interpretation of  $f_{n=N}$  as a sum over connected fatgraphs as the logarithm of (1.5), each connected fatgraph must have at least one  $M$  and one  $R$ -vertex, resulting in a factor  $ab$ . For  $n \neq N$ , an analogous expansion holds, and the same conclusion applies. In this way, we recover the results (2.63). Note that the step of discrete integration is simpler here than in (3.13), where the two discrete integrations wrt  $n$  actually involved the determination of  $\omega_{n,k}$  for all  $n = 0, 1, 2, \dots$ . Here this step can be done in general without having to compute explicitly for all (integer) values  $a, b = 0, 1, 2, \dots$ . It is much more economical when implemented on a computer.

### 3.3. Large $N$ limit

In the planar  $N \rightarrow \infty$  limit, we still set  $z = n/N$ . We also assume the leading behavior  $f_n \sim N^2 f(z; p, q, t)$ , so that the large  $N$  free energy (2.65) reads  $F_0(p, q, t) = f(1; p, q, t)$ . The above difference operators (3.11) become differential operators, namely

$$\delta_n \rightarrow \frac{1}{N} \partial_z \quad \delta_a \rightarrow \frac{1}{N} \partial_p \quad \delta_b \rightarrow \frac{1}{N} \partial_q \quad (3.14)$$

and the Hirota equation (3.12) becomes a partial differential equation for the function  $f \equiv f(z; p, q, t)$

$$\partial_p \partial_q f = -\text{Log}(1 - tze^{\partial_z(\partial_z + \partial_p + \partial_q)f}) \quad (3.15)$$

Note that  $f$  is not determined entirely by this equation, in particular we may add to  $f$  any function of  $t$ . However, as mentioned earlier,  $f$  has  $pq$  in factor, which implies that  $\partial_p f(z; 0, 0, t) = \partial_q f(z; 0, 0, t) = 0$ . We therefore solve an equation of the form  $\partial_p \partial_q f(p, q) = g(p, q)$  as  $f = \int_0^p dx \int_0^q dy g(x, y)$  (we will actually implement this on the coefficients  $\omega_k(z; p, q)$  of the expansion (2.60), which are polynomials of  $p, q$  for all  $k$ ). Substituting the small  $t$  expansion (2.71) into (3.15), and integrating wrt  $p$  and  $q$ , we finally obtain

$$\sum_{n \geq 0} \omega_n(z; p, q) t^n = - \int_0^p dx \int_0^q dy \text{Log}(1 - tze^{\sum_{k \geq 0} \partial_z(\partial_z + \partial_x + \partial_y) \omega_k(z; x, y) t^k}) \quad (3.16)$$

which is a recursion relation for  $\omega_n$ . With the initial condition  $\omega_0(z; p, q) = 0$ , we recover the result (2.84). Note that both (2.68) and (3.15) are partial differential equations determining  $f$  up to initial conditions, they must therefore be compatible, a highly non-trivial fact considering their very different forms.

Remarkably, combining both equations (2.68) and (3.15) allows us to solve the case  $p = q = z$  explicitly, with the following result for the reduced free energy<sup>2</sup>

$$\begin{aligned} F(z, t) &= f(z; z, z, t) \\ &= 3z^2 \sum_{k=1}^{\infty} (2tz)^k \frac{(2k-1)!}{k!(k+2)!} \end{aligned} \quad (3.17)$$

as we will prove now. The  $(p, q, z)$  symmetry (2.86) of  $f$  implies that

$$\partial_z(\partial_z + \partial_p + \partial_q)f(z; z, z, t) = (\partial_z^2 + 2\partial_p \partial_q)f(z; z, z, t) = \frac{1}{3}\partial_z^2 F(z, t) \quad (3.18)$$

Now using the equations (2.68) and (3.15) in the form

$$\begin{aligned} \partial_z^2 f &= \text{Log}\left(1 + \frac{t}{z}(t\partial_t)^2 f\right) \\ \partial_p \partial_q f &= -\text{Log}(1 - tze^{\partial_z(\partial_z + \partial_p + \partial_q)f}) \end{aligned} \quad (3.19)$$

---

<sup>2</sup> This result coincides with the generating function of planar rooted bicubic maps of  $2n$  vertices [14].

and adding twice the second line to the first one, we get, using (3.18), the equation satisfied by  $F(z, t)$

$$\begin{aligned} \frac{1}{3}\partial_z^2 F(z, t) &= \text{Log}\left(1 + \frac{t}{z}(t\partial_t)^2 F(z, t)\right) \\ &\quad - 2\text{Log}\left(1 - zte^{\frac{1}{3}\partial_z^2 F(z, t)}\right) \end{aligned} \quad (3.20)$$

To check that (3.17) is a solution of this equation, we can relate the expression

$$\frac{1}{3}\partial_z^2 F(z, t) = \sum_{k=1}^{\infty} (2tz)^k \frac{(2k-1)!}{(k!)^2} \quad (3.21)$$

to the generating function of the celebrated Catalan numbers

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{k \geq 0} c_k x^k \quad (3.22)$$

with

$$c_k = \frac{(2k)!}{k!(k+1)!} \quad (3.23)$$

Setting  $x = 2zt$ , we may reexpress the result (3.21) as

$$\begin{aligned} \frac{1}{3}\partial_z^2 F(z, t) &= \int_0^x \frac{dy}{2y} \left( \frac{d}{dy}(yC(y)) - 1 \right) \\ &= \int_0^x \frac{dy}{2y} \left( \frac{1}{\sqrt{1-4y}} - 1 \right) \\ &= \frac{1}{2}\text{Log} \left( \frac{1 - \sqrt{1-4x}}{1 + \sqrt{1-4x}} \right) - \frac{1}{2}\text{Log } x \\ &= \text{Log } C(x) \end{aligned} \quad (3.24)$$

Exponentiating both sides of (3.20) and using (3.24), our result will hold provided

$$C(x) \left(1 - \frac{x}{2}C(x)\right)^2 = 1 + \frac{t}{z}(t\partial_t)^2 F(z, t) \quad (3.25)$$

with  $x = 2zt$  as above. But thanks to the quadratic relation  $xC(x)^2 = C(x) - 1$  satisfied by (3.22), we may rewrite the lhs of (3.25) as

$$\begin{aligned} 1 + \frac{x^2}{4}C(x)^3 &= 1 + \frac{x}{4}C(x)(C(x) - 1) \\ &= 1 + \frac{1}{4}((1-x)C(x) - 1) \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{4}(c_k - c_{k-1}) \\ &= 1 + \frac{3}{4} \sum_{k=0}^{\infty} (2tz)^{k+1} \frac{(2k)!}{(k-1)!(k+2)!} \\ &= 1 + 3tz(t\partial_t)^2 \sum_{k=0}^{\infty} (2tz)^k \frac{(2k-1)!}{k!(k+2)!} \end{aligned} \quad (3.26)$$



This proves (3.25) for  $F(z, t)$  as in (3.17). Since (3.17) also satisfies the correct initial conditions  $F(0, t) = f(0; 0, 0, t) = 0$  and  $\partial_z F(0, t) = 3\partial_z f(0; 0, 0, t) = 0$ , it is therefore identified as the desired solution for  $p = q = z$ . Note that, from the convergence radius of the series (3.17), this solution has a critical point at

$$t_*(z, z, z) = \frac{1}{8z} \quad (3.27)$$

Actually, the generating function at  $p = q = z = 1$  for bi-colored fatgraphs with a *marked* edge reads (see Appendix A for a similar study)

$$(t\partial_t)F(1, t) = \sum_{k=1}^{\infty} t^k \nu_k \quad (3.28)$$

where

$$\nu_k = \frac{3}{2} 2^k \frac{c_k}{k+2} \quad (3.29)$$

Remarkably, the numbers  $\nu_k$  have already emerged in the context of arches and meanders [16], within the framework of the Temperley-Lieb algebra. A more detailed study would indeed lead to the identification between certain pairs of arch configurations of order  $k$  (with  $k$  arches) and bi-colored diagrams with  $k$  edges, thus yielding a different interpretation of the result (3.29).

## 4. Direct expansion

### 4.1. The partition function as a sum over integers

We start from the formula (3.5) for the  $n \times n$  determinant  $\phi_n(a, b)$ . For generic (non-integer) values of  $a$  and  $b$ , we may expand

$$(1-x)^{i-a-1} = \sum_{k \geq 0} \frac{\Gamma(k+a-i+1)}{\Gamma(1+a-i)} \frac{x^k}{k!} \quad (4.1)$$

and similarly for  $(1-y)^{j-b-1}$ . Using the formal integral

$$\int dx dy x^\alpha y^\beta e^{-Nxy/t} = \alpha! \delta_{\alpha, \beta} (t/N)^{\alpha+1} \quad (4.2)$$

for any integer  $\alpha$ , we get

$$\begin{aligned}\phi_n(a, b) &= \det \left[ \sum_{k \geq 0} \frac{1}{k!} \frac{\Gamma(k + a - i + 1)}{\Gamma(1 + a - i)} \frac{\Gamma(k + b - j + 1)}{\Gamma(1 + b - j)} \left(\frac{t}{N}\right)^{k+1} \right]_{1 \leq i, j \leq n} \\ &= \sum_{k_1, \dots, k_n \geq 0} \prod_{i=1}^n \frac{1}{k_i!} \frac{\Gamma(k_i + a - i + 1)}{\Gamma(1 + a - i)} \frac{(t/N)^{k_i+1}}{\Gamma(1 + b - i)} \det [\Gamma(k_i + b - j + 1)]_{1 \leq i, j \leq n}\end{aligned}\quad (4.3)$$

where we have used the multilinearity of the determinant to extract line by line the summations over  $k$ 's. Factoring  $\Gamma(k_i + b - n + 1)$  out of each line (number  $i$ ) of the remaining determinant, we are left with a determinant of the form

$$\det [(k_i + b - j)(k_i + b - j - 1) \dots (k_i + b - n + 1)] = \det [q_{n-j}(k_i)] \quad (4.4)$$

where the polynomials  $q_m(x) = x^m + \text{lower degree}$  are monic. As in (2.5), we may identify (4.4) as the Vandermonde determinant  $\Delta(k)$  of the diagonal matrix  $k = \text{diag}(k_1, k_2, \dots, k_n)$ . This yields

$$\phi_n(a, b) = \sum_{k_1, \dots, k_n \geq 0} \prod_{i=1}^n (t/N)^{k_i+1} \frac{1}{k_i!} \frac{\Gamma(k_i + a - i + 1)}{\Gamma(1 + a - i)} \frac{\Gamma(k_i + b - n + 1)}{\Gamma(1 + b - i)} \Delta(k) \quad (4.5)$$

Using the antisymmetry of  $\Delta(k)$ , we may symmetrize the above expression into

$$\begin{aligned}\phi_n(a, b) &= \frac{1}{n!} \sum_{k_1, \dots, k_n \geq 0} \prod_{i=1}^n \frac{1}{k_i!} \frac{(t/N)^{k_i+1}}{\Gamma(1 + a - i)} \frac{\Gamma(k_i + b - n + 1)}{\Gamma(1 + b - i)} \\ &\quad \times \Delta(k) \det [\Gamma(k_i + a - j + 1)]_{1 \leq i, j \leq n}\end{aligned}\quad (4.6)$$

Factoring again  $\Gamma(k_i + a - n + 1)$  for each line (number  $i$ ) of the last determinant, and repeating the above trick, we finally get

$$\phi_n(a, b) = \frac{1}{n!} \sum_{k_1, \dots, k_n \geq 0} \Delta(k)^2 \prod_{i=1}^n (t/N)^{k_i+1} \frac{1}{k_i!} \frac{\Gamma(k_i + a - n + 1)}{\Gamma(1 + a - i)} \frac{\Gamma(k_i + b - n + 1)}{\Gamma(1 + b - i)} \quad (4.7)$$

or, using (3.4),

$$Z_n(a, b) = \sum_{k_1, \dots, k_n \geq 0} \Delta(k)^2 \prod_{i=1}^n \frac{(t/N)^{k_i+1-i}}{i!} \frac{1}{k_i!} \frac{\Gamma(k_i + a - n + 1)}{\Gamma(1 + a - i)} \frac{\Gamma(k_i + b - n + 1)}{\Gamma(1 + b - i)} \quad (4.8)$$

The expression summed over is symmetric in the  $k_i$ 's, hence we may further reduce the sum to only strictly increasing sequences of  $k_i$ 's, namely

$$Z_n(a, b) = \sum_{0 \leq k_1 < k_2 < \dots < k_n} \Delta(k)^2 \prod_{i=1}^n \frac{(t/N)^{k_i - (i-1)}}{(i-1)!} \frac{1}{k_i!} \frac{\Gamma(k_i + a - n + 1)}{\Gamma(1 + a - i)} \frac{\Gamma(k_i + b - n + 1)}{\Gamma(1 + b - i)} \quad (4.9)$$

This expression leads to a straightforward expansion of  $Z_n$  in powers of  $t$ .

The leading  $t^0$  term corresponds to  $k_i = i - 1$ , which we refer to as the *fundamental* (or *groundstate*) configuration of the  $k_i$ 's. It has the contribution 1. The term of order one in  $t$  is obtained by creating the excited state where  $k_i$  are unchanged for  $i = 1, \dots, n - 1$  and  $k_n = n - 1 \rightarrow k_n = n$ . Its contribution is easily computed as  $nabt/N$ . We may think of the power of  $t$  as the energy of the excited state (1 here). In general the term of order  $t^k$  will be obtained by a number of excitations of the form  $k_i = i - 1 + \delta k_i$ , which respect the strict ordering of the  $k$ 's (i.e.,  $\delta k_i \leq \delta k_{i+1}$  for all  $i$ ), and with a total energy  $k = \sum_i \delta k_i$ . This makes the expansion of  $Z_n(a, b)$  quite explicit.

#### 4.2. Explicit Expansions

As an illustration of the use of (4.9), let us re-work the example  $a = -1$ ,  $b$  arbitrary, of (2.56), by using the expansion (4.9). The terms  $\Gamma(1 + a - i) = \Gamma(-i)$ ,  $i = 1, 2, \dots, n$ , in the denominator may cause the contribution to vanish, *unless* they are counterbalanced by terms of the form  $\Gamma(-j)$ ,  $j \geq 0$  coming from the numerator. Hence, the only non-zero contributions to the sum (4.9) arise when  $k_i + a - n + 1 = k_i - n \leq 0$ , for all  $i$ . The  $k$ 's being strictly ordered, they are either in the fundamental configuration

$$K_0 = (k_1, k_2, k_3, \dots, k_n) = (0, 1, 2, 3, \dots, n - 1) \quad (4.10)$$

or in one of the following  $n - 1$  excited states

$$K_j = (0, 1, \dots, j - 2, j - 1, j + 1, j + 2, \dots, n) \quad (4.11)$$

for  $j = 1, 2, \dots, n - 1$ . Assembling the contributions from the configurations  $K_0, K_1, \dots, K_{n-1}$ , we get

$$Z_n(-1, b) = 1 + \sum_{j=1}^{n-1} \left(\frac{t}{N}\right)^{n-j} \binom{n}{j} \frac{\Gamma(1 + b)}{\Gamma(1 + b - n + j)} \quad (4.12)$$

which is equivalent to (2.55).

The example  $a = 1$  is also instructive. The denominators  $\Gamma(2 - i)$  cause trouble for  $i = 2, 3, \dots, n$ . Hence one of the  $k$ 's may be chosen freely, while all the others must satisfy  $k_i + a - n + 1 = k_i - (n - 2) \leq 0$ . The strict ordering of the  $k$ 's imposes the choice

$$(k_1, k_2, \dots, k_{n-1}) = (0, 1, 2, \dots, n-2) \quad \text{and} \quad k_n \geq n-1 \quad (4.13)$$

This results in an infinity of possible excitations  $k_n = n - 1 + k$  above the fundamental  $k = 0$ . These contribute for

$$Z_n(1, b) = \sum_{k=0}^{\infty} \left( \frac{t}{N} \right)^k \binom{n+k-1}{k} \frac{\Gamma(k+b)}{\Gamma(b)} \quad (4.14)$$

which is equivalent to (2.58).

The formula (4.9) permits also to prove directly the previously-mentioned  $(a, b, n)$  symmetry of  $Z_n(a, b) = Z_a(n, b)$  for  $a = m$  any positive integer. Indeed, for  $a = 1$ , if we compute  $Z_1(n, b)$  using (4.9), we find that  $k_1 = k$  may take the fundamental value  $k = 0$  or any excitation  $k > 0$ , resulting in the same sum as (4.14), therefore

$$Z_1(n, b) = Z_n(1, b) \quad (4.15)$$

More generally, let us evaluate  $Z_n(m, b)$  using (4.9). The term  $\Gamma(1 + a - i) = \Gamma(m + 1 - i)$  in the denominator causes problems for  $i \geq m + 1$  only. Assuming that  $n > m$ , this forces  $(n - m)$  of the  $k$ 's to satisfy  $k_i + a - n + 1 = k_i - (n - m - 1) \leq 0$ . These are necessarily

$$(k_1, k_2, \dots, k_{n-m}) = (0, 1, \dots, n - m - 1) \quad (4.16)$$

as the sequence is strictly increasing. The remaining  $k$ 's may be in either the fundamental state or any excited state above it, namely

$$n - m \leq k_{n-m+1} < k_{n-m+2} < \dots < k_n \quad (4.17)$$

If we look instead at  $Z_m(n, b)$ , there are no bad denominators, and all the excitations of the  $m$   $k$ 's are permitted, namely any

$$0 \leq k'_1 < k'_2 < \dots < k'_m \quad (4.18)$$

The two sets of excitations (4.17) and (4.18) can be mapped onto each other by writing

$$k_j = n - m + k'_{j+m-n} \quad \forall j = n - m + 1, n - m + 2, \dots, n \quad (4.19)$$

and it is a straightforward exercise to check that their contributions in (4.9) are identical. This proves the symmetry  $Z_n(m, b) = Z_m(n, b)$  for all  $m, n, b$ .

### 4.3. The large $N$ limit: general solution

When  $N$  is large, with  $n = zN$ , we expect the sum (4.8) to be dominated by integers of the form  $k_i = N\alpha_i$ , where  $\alpha_i$  are real numbers. The index  $i$  itself ranging from 1 to  $n = zN$ , we may let it scale as  $i = Ns$ ,  $s$  ranging from 0 to  $z$ . The sum (4.8) may therefore be approximated by the multiple integral

$$Z_n(a, b) \sim N^n \int_0^\infty d\alpha_1 \dots d\alpha_n e^{N^2 S(\alpha_1, \dots, \alpha_n; p, q, z)} \quad (4.20)$$

where we have identified the effective action

$$\begin{aligned} S(\alpha_1, \dots, \alpha_n; p, q, z) = & \frac{1}{N} \sum_{i=1}^n [(\alpha_i + p - z)(\text{Log}(\alpha_i + p - z) - 1) \\ & + (\alpha_i + q - z)(\text{Log}(\alpha_i + q - z) - 1) - \alpha_i(\text{Log}(\alpha_i) - 1) + \alpha_i \text{Log } t] \\ & + \frac{1}{N^2} \sum_{1 \leq i \neq j \leq n} \text{Log}|\alpha_i - \alpha_j| - \int_0^z ds [(p - s)(\text{Log}(p - s) - 1) \\ & + (q - s)(\text{Log}(q - s) - 1) + s(\text{Log}(s) - 1) + s \text{Log } t] \end{aligned} \quad (4.21)$$

by use of the Stirling formula. The integral (4.20) is dominated by the saddle-point  $\partial S / \partial \alpha_i = 0$ , namely

$$\text{Log} \left( t \frac{(\alpha_i + p - z)(\alpha_i + q - z)}{\alpha_i} \right) = \frac{2}{N} \sum_{j \neq i} \frac{1}{\alpha_j - \alpha_i} \quad (4.22)$$

for  $i = 1, 2, \dots, n$ . Introducing the large  $N$  density

$$\rho(\alpha) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^n \delta(\alpha - \alpha_i) \quad (4.23)$$

with the normalization

$$\int d\alpha \rho(\alpha) = z \quad (4.24)$$

we get the limiting equation

$$\boxed{\text{Log} \left( t \frac{(\alpha + p - z)(\alpha + q - z)}{\alpha} \right) = 2P \int d\beta \frac{\rho(\beta)}{\beta - \alpha}} \quad (4.25)$$

where the symbol  $P$  stands for the principal value. As usual, we introduce the resolvent

$$\omega(\alpha) = \int d\beta \frac{\rho(\beta)}{\alpha - \beta} \quad (4.26)$$

for any complex number  $\alpha$ . In particular,  $\omega(\alpha)$  behaves for large  $|\alpha|$  as

$$\omega(\alpha) \sim \frac{1}{\alpha} \int \rho(\beta) d\beta = \frac{z}{\alpha} \quad (4.27)$$

by use of the normalization (4.24). Moreover, the saddle-point equation (4.25) may be written as

$$\lim_{\epsilon \rightarrow 0^+} \omega(\alpha + i\epsilon) + \omega(\alpha - i\epsilon) = -\text{Log} \left( t \frac{(\alpha + p - z)(\alpha + q - z)}{\alpha} \right) \quad (4.28)$$

If we can solve this equation, we can deduce the large  $N$  value of the free energy from the effective action (4.21). Indeed, writing  $S(\alpha_i^*; p, q, z; t) = f(p, q, z; t)$  the saddle-point action, we may compute

$$\begin{aligned} t\partial_t f &= \sum_i \partial_{\alpha_i} S(\alpha_i) t \frac{d\alpha_i}{dt} \Big|_{\alpha_i = \alpha_i^*} + t\partial_t S \\ &= \int \beta \rho(\beta) d\beta - \frac{z^2}{2} \end{aligned} \quad (4.29)$$

where we have used (4.22) at the saddle-point  $\alpha_i = \alpha_i^*$  and the definition (4.23) of the limiting density  $\rho$ . In terms of the resolvent

$$\omega(\alpha) = \sum_{i=1}^{\infty} \frac{\Omega_i}{\alpha^i} \quad (4.30)$$

this simply reads

$$t\partial_t f(p, q, z; t) = \Omega_2 - \frac{z^2}{2} \quad (4.31)$$

To solve (4.28), let us perform the change of variables

$$\begin{aligned} \alpha &= z - r - 2\delta \cosh \phi \\ p &= r + 2\delta \cosh \phi_1 \\ q &= r + 2\delta \cosh \phi_2 \\ z &= r + 2\delta \cosh \phi_3 \end{aligned} \quad (4.32)$$

where  $r$  and  $\delta$  are two free parameters, to be fixed later. We will take the variable  $\phi = i\varphi$ ,  $\varphi \in [0, \pi]$ , to be purely imaginary on the support of the distribution  $\rho$ , so that  $z - r$  and  $4\delta$  measure the center and width of this support. We define

$$\begin{aligned} u &= e^{-\phi}, & T &= \tanh(\phi/2) = \frac{1-u}{1+u} \\ u_i &= e^{-\phi_i}, & T_i &= \tanh(\phi_i/2) = \frac{1-u_i}{1+u_i}, & i &= 1, 2, 3 \end{aligned} \quad (4.33)$$

Then we have

$$\frac{t}{\alpha}(\alpha + p - z)(\alpha + q - z) = \delta t \frac{4(T_1^2 - T^2)(T_2^2 - T^2)(1 - T_3^2)}{(1 - T^2)(1 - T_1^2)(1 - T_2^2)(T_3^2 - T^2)} \quad (4.34)$$

Since on the support of  $\rho$ ,  $T = i \tan(\varphi/2)$  satisfies  $T^* = -T$ , the saddle point equation (4.28) has the following solution

$$\omega(\alpha) = -\text{Log} \left( \frac{2(T_1 + T)(T_2 + T)(1 - T_3)}{(1 + T)(1 + T_1)(1 + T_2)(T - T_3)} \right) \quad (4.35)$$

for the resolvent provided we take  $\delta$  such that

$$\delta t = \frac{(1 - T_1)(1 - T_2)(1 - T_3)}{(1 + T_1)(1 + T_2)(1 + T_3)} = u_1 u_2 u_3 \quad (4.36)$$

Moreover, the large  $\alpha$  behavior (4.27) imposes that

$$\omega(\alpha) \sim u(u_1 + u_2 - \frac{1}{u_3}) = -\frac{zu}{\delta} \quad (4.37)$$

namely that

$$u_1 u_2 u_3 \left( \frac{1}{u_3} - u_1 - u_2 \right) = zt \quad (4.38)$$

Using the third line of (4.32), this fixes the value of  $r$  to

$$r = -\delta(u_1 + u_2 + u_3) \quad (4.39)$$

Substituting this value into the second and third lines of (4.32), and using the value (4.36) for  $\delta$ , we get

$$\begin{aligned} u_1 u_2 u_3 \left( \frac{1}{u_1} - u_2 - u_3 \right) &= pt \\ u_1 u_2 u_3 \left( \frac{1}{u_2} - u_1 - u_3 \right) &= qt \end{aligned} \quad (4.40)$$

Introducing the new variables

$$U_1 = u_2 u_3 \quad U_2 = u_1 u_3 \quad U_3 = u_1 u_2 \quad (4.41)$$

we may recast (4.38) and (4.40) into

$$\begin{aligned} U_1(1 - U_2 - U_3) &= pt \\ U_2(1 - U_1 - U_3) &= qt \\ U_3(1 - U_1 - U_2) &= zt \end{aligned} \quad (4.42)$$

By eliminating  $U_2$  and  $U_3$  as

$$\begin{aligned} U_2 &= \frac{1}{2}\left(1 - \frac{pt}{U_1} + \frac{(q-z)t}{1-U_1}\right) \\ U_3 &= \frac{1}{2}\left(1 - \frac{pt}{U_1} - \frac{(q-z)t}{1-U_1}\right) \end{aligned} \quad (4.43)$$

this reduces to the following 5-th order equation for  $U_1 \equiv U_1(p, q, z; t)$

$$\boxed{U_1^2(1-U_1)^2(1-2U_1+2(p-q-z)t) = t^2((1-U_1)^2p^2 - U_1^2(z-q)^2)} \quad (4.44)$$

We have to retain the unique solution behaving for small  $t$  as  $U_1 \sim pt$  (this is the unique small  $t$  behavior compatible with Eq.(4.49) below). The other  $U$ 's are obtained by exchanging the values of  $p, q, z$ , namely

$$\begin{aligned} U_2(p, q, z; t) &= U_1(q, p, z; t) \\ U_3(p, q, z; t) &= U_1(z, q, p; t) \end{aligned} \quad (4.45)$$

We then have to plug these solutions back into

$$\begin{aligned} \delta &= \frac{1}{t}\sqrt{U_1U_2U_3} \\ r &= -\frac{1}{t}(U_1U_2 + U_2U_3 + U_1U_3) \\ u_i &= \frac{\sqrt{U_1U_2U_3}}{U_i} \quad i = 1, 2, 3 \end{aligned} \quad (4.46)$$

For instance, the density associated to the resolvent (4.35) reads

$$\begin{aligned} \rho(\alpha) &= \frac{1}{2i\pi} \lim_{\epsilon \rightarrow 0^+} \omega(\alpha + i\epsilon) - \omega(\alpha - i\epsilon) \\ &= \frac{1}{2\pi} (\varphi - 2(\Psi_1 + \Psi_2 + \Psi_3)) \end{aligned} \quad (4.47)$$

with  $\alpha = z - r - 2\delta\cos(\varphi)$ , and we have introduced the quantities

$$\Psi_i = \tan^{-1}\left(\frac{\tan(\frac{\varphi}{2})}{T_i}\right) \quad (4.48)$$

The free energy is easily obtained from (4.31) by expanding the resolvent (4.35) up to the second order in  $1/\alpha$ . We find

$$\boxed{t\partial_t f(p, q, z; t) = \Omega_2 - \frac{z^2}{2} = \frac{U_1U_2U_3}{t^2}(1 - U_1 - U_2 - U_3)} \quad (4.49)$$

Note that this is explicitly symmetric in  $p, q, z$  as expected.



#### 4.4. Particular cases

In the passage (4.43) from (4.42) to (4.44), we implicitly assumed that  $U_1$  was different from 0 and 1. Let us examine these two cases in more details.

If  $U_1 = 0$ , we read from (4.42) that  $p = 0$ . Conversely, if  $p = 0$ , we deduce from (4.42) that either  $U_1 = 0$  identically or  $U_2 + U_3 = 1$  identically. The latter solution is however not compatible with the requirement that  $U_2 + U_3 \sim (q + z)t$  at small  $t$  and should be discarded. Setting  $U_1 = 0$  in (4.42) leads to

$$\begin{aligned} U_2 - U_2 U_3 &= qt \\ U_3 - U_2 U_3 &= zt \end{aligned} \tag{4.50}$$

Since  $\delta = 0$  in this limit, the support of the distribution  $\rho$  reduces to one point located at

$$\alpha = \frac{U_2 U_3}{t} \tag{4.51}$$

which, using (4.50), is solution of

$$t(\alpha + q)(\alpha + z) = \alpha \tag{4.52}$$

We thus recover the saddle-point result (2.75) with  $p \leftrightarrow z$ .

If  $U_1 = 1$ , we read from (4.42) that  $q = z$ . Conversely, if  $q = z$ , we deduce from (4.42) that either  $U_1 = 1$  identically or  $U_2 = U_3$  identically. From the requirement that  $U_1 \sim pt$  for small  $t$ , we see that it is now the latter solution which has to be kept while the solution  $U_1 = 1$  is simply unphysical. Let us nevertheless study this particular case  $q = z$ , now described by its correct solution  $U_2 = U_3$ . We now get a third order equation for  $U_1$

$$Q(U_1, t) = U_1^2(1 - 2U_1 + 2(p - 2z)t) - p^2 t^2 = 0 \tag{4.53}$$

This fixes  $U_1$  as a function of  $t$ . Writing  $\partial_t U_1 = -\partial_t Q(U_1, t)/\partial_u Q(U_1, t)$ , we find a first order critical point  $t = t_*(p, z, z)$ ,  $U_1 = u_*$ , whenever  $\partial_u Q(u_*, t_*) = 0$ , hence

$$\partial_u Q(u_*, t_*) = 2u_*(1 - 2u_* + 2(p - 2z)t_*) - 2u_*^2 = 0 \tag{4.54}$$

By combining with (4.53), we get  $u_*^3 = (pt_*)^2$ . As  $u_* > 0$ , it is convenient to introduce the parametrization

$$u_* = \frac{1}{4 \cos^2 \frac{\theta}{3}} \tag{4.55}$$

where

$$\begin{aligned} \theta &\geq 0 \quad \text{when} \quad \frac{1}{4} \leq u_* \\ \theta &= i\varphi, \quad \varphi \geq 0 \quad \text{when} \quad 0 \leq u_* \leq \frac{1}{4} \end{aligned} \quad (4.56)$$

The critical point is then

$$t_*(p, z, z) = \frac{\epsilon}{8p \cos^3 \frac{\theta}{3}} \quad (4.57)$$

with  $\epsilon = \pm 1$ . The values of  $\theta$  and  $\epsilon$  are fixed by the ratio

$$\begin{aligned} \frac{z}{p} = \frac{q}{p} &= \frac{1 + \epsilon \cos \theta}{2} \\ &= \begin{cases} \cos^2 \frac{\theta}{2} & \text{if } \epsilon = 1 \\ \sin^2 \frac{\theta}{2} & \text{if } \epsilon = -1 \end{cases} \end{aligned} \quad (4.58)$$

For  $z/p > 1$ , (4.58) has exactly one solution corresponding to  $\epsilon = 1$ ,  $z/p = \cosh^2(\varphi/2)$ , and we find a critical point at  $t_* = 1/(8p \cosh^3(\varphi/3))$ . We thus have

$$t_*(p, z, z) = \frac{1}{8p \cosh^3 \left( \frac{2}{3} \operatorname{arccosh} \sqrt{\frac{z}{p}} \right)} \quad (4.59)$$

For  $z/p < 0$ , (4.58) gives also exactly one solution  $(u_*, t_*)$  with now  $\epsilon = -1$ . However, this point is not on the correct branch of the solution of (4.53), i.e. that satisfying  $U_1 \sim pt$  at small  $t$ . This is easily checked by solving explicitly (4.53), which is quadratic in  $t$  in the form  $t(U_1)$  rather than  $U_1(t)$ , and choosing the solution with the correct small  $t$  behavior. We thus deduce that there is no critical point ( $R = +\infty$ ) for  $z/p < 0$ .

For  $0 \leq z/p \leq 1$  (4.53) has an infinite number of solutions of the form  $\epsilon = 1$ ,  $\theta = \theta_0 + 2k\pi$ ,  $\theta = (2k+1)\pi - \theta_0$  and  $\epsilon = -1$ ,  $\theta = \theta_0 + (2k+1)\pi$ ,  $\theta = (2k+2)\pi - \theta_0$  with  $\theta_0 \in [0, \pi[$  and  $k$  a non negative integer. This leads to exactly three candidates for the critical point  $t_* = 1/(8p \cos^3(\theta_0/3))$ ,  $t_* = 1/(8p \cos^3(\theta_0/3 + 2\pi/3))$  and  $t_* = 1/(8p \cos^3(\theta_0/3 + 4\pi/3))$ . Here again, it is easy to check that the critical point attached to the correct branch is the first one, namely

$$t_*(p, z, z) = \frac{1}{8p \cos^3 \left( \frac{2}{3} \arccos \sqrt{\frac{z}{p}} \right)} \quad (4.60)$$

where the arccos is taken in the range  $[0, \pi[$ .

When  $p = q = z$ , we recover the critical value  $t_*(z, z, z) = 1/8z$  of (3.27) by setting  $\theta_0 = 0$ , with moreover  $u_* = 1/4$ . This case  $p = q = z$  can be solved more directly by

noting that, from (4.42), we then have  $U_1 = U_2 = U_3 \equiv U$ , where  $U$  is the solution of a quadratic equation  $U(1 - 2U) = zt$ , namely

$$U = \frac{1}{4}(1 - \sqrt{1 - 8zt}) = ztC(2zt) \quad (4.61)$$

The corresponding free energy reads

$$t\partial_t F(z; t) = \frac{U^3}{t^2}(1 - 3U) = \frac{z}{8t}((8zt - 1)C(2zt) + 1 - 6zt) \quad (4.62)$$

which coincides with our previous result (3.28)(3.29).

#### 4.5. Critical Points

The solution  $U_1$  to (4.44) such that  $U_1 \sim pt$  when  $t \rightarrow 0$  will in general have a convergent series expansion until we reach a critical point  $t = t_*(p, q, z)$ , corresponding to the convergence radius  $R = t_*$ . To determine this point, we write the (finite) Taylor series of the polynomial

$$P(u, t) = u^2(1 - u)^2(1 - 2u + 2(p - q - z)t) - (1 - u)^2p^2t^2 + u^2(q - z)^2t^2 \quad (4.63)$$

namely

$$\begin{aligned} P(u, t) &= P(u_*, t_*) + (u - u_*)\partial_u P(u_*, t_*) + \frac{(u - u_*)^2}{2}\partial_u^2 P(u_*, t_*) \\ &\quad + (t - t_*)\partial_t P(u_*, t_*) + \dots \end{aligned} \quad (4.64)$$

Imposing that  $P$  vanishes, we get a first order critical point

$$(u_* - u) \sim (t_* - t)^{1/2} \quad (4.65)$$

by demanding that

$$P(u_*, t_*) = \partial_u P(u_*, t_*) = 0 \quad (4.66)$$

while

$$\partial_u^2 P(u_*, t_*) \neq 0 \quad \partial_t P(u_*, t_*) \neq 0 \quad (4.67)$$

This means that the polynomial  $P(u, t_*)$  takes the form

$$P(u, t_*) = (u - u_*)^2(-2u^3 + au^2 + bu + c) \quad (4.68)$$

for some numbers  $a, b, c$  to be determined. Introducing the parameters

$$\alpha = 2(p - q - z)t_* \quad \beta = p^2 t_*^2 \quad \gamma = (q - z)^2 t_*^2 \quad (4.69)$$

and identifying (4.63) with (4.68), we get

$$\begin{aligned} a + 4u_* &= \alpha + 5 \\ b - 2au_* - 2u_*^2 &= -2(\alpha + 2) \\ c - 2bu_* + au_*^2 &= 1 + \alpha - \beta + \gamma \\ bu_*^2 - 2cu_* &= 2\beta \\ cu_*^2 &= -\beta \end{aligned} \quad (4.70)$$

Eliminating  $\beta$  from the last two lines of (4.70), and  $\alpha$  from the first two, we find

$$\begin{aligned} b &= 2c \frac{1 - u_*}{u_*} \\ a &= 3 - u_* - \frac{c}{u_*} \end{aligned} \quad (4.71)$$

Here we can assume that  $u_* \neq 0, 1$ , since the situation where  $U_1 = 0$  or  $1$  has been treated in the previous section. The equations (4.71) translate into the following parametrization of  $\alpha, \beta, \gamma$  of (4.69)

$$\begin{aligned} \alpha &= 2(p - q - z)t_* = 3u_* - 2 + \xi \\ \beta &= p^2 t_*^2 = \xi u_*^3 \\ \gamma &= (q - z)^2 t_*^2 = (1 - u_*)^3 (1 - \xi) \end{aligned} \quad (4.72)$$

where we have set  $c = -\xi u_*$ . This may be viewed as a parametric curve  $(pt_*, qt_*, zt_*) = h(u_*, \xi)$ , where  $u_*$  and  $\xi$  are real parameters, with  $u_* \xi \geq 0$  and  $(1 - u_*)(1 - \xi) \geq 0$ . This in turn permits to express for instance

$$\boxed{t_* = \frac{\epsilon}{p} \sqrt{\xi u_*^3}} \quad (4.73)$$

where  $\epsilon = \text{sgn}(pt_*)$ , as an implicit function of the ratios  $q/p$  and  $z/p$ , where

$$\boxed{\begin{aligned} \frac{q}{p} &= \frac{1}{2} \left( 1 + \epsilon' \frac{1 - u_*}{u_*} \sqrt{\frac{(1 - u_*)(1 - \xi)}{u_* \xi}} - \epsilon \frac{3u_* - 2 + \xi}{2u_* \sqrt{\xi u_*}} \right) \\ \frac{z}{p} &= \frac{1}{2} \left( 1 - \epsilon' \frac{1 - u_*}{u_*} \sqrt{\frac{(1 - u_*)(1 - \xi)}{u_* \xi}} - \epsilon \frac{3u_* - 2 + \xi}{2u_* \sqrt{\xi u_*}} \right) \end{aligned}} \quad (4.74)$$

and where  $\epsilon' = \text{sgn}(p(q-z))$ . Note that the transformation  $(\xi, u_*) \rightarrow (1-\xi, 1-u_*)$  simply amounts to

$$\begin{aligned} p &\rightarrow \epsilon'(q-z) \\ q &\rightarrow \frac{1+\epsilon'}{2}p + \frac{\epsilon'-1}{2}q - \epsilon'z \\ z &\rightarrow \frac{1-\epsilon'}{2}p + \frac{\epsilon'+1}{2}q - \epsilon'z \end{aligned} \quad (4.75)$$

We must now check the validity of this critical point, by making sure that (4.67) is satisfied. At the critical point, we find

$$\begin{aligned} \partial_t P(u_*, t_*) &= \frac{1}{t_*} u_*^2 (1-u_*)^2 (u_* - \xi) \\ \partial_u^2 P(u_*, t_*) &= 6u_*(1-u_*)(u_* - \xi) \end{aligned} \quad (4.76)$$

Provided we take  $u_* \neq 0, 1, \xi$ , we see that (4.67) will hold. As mentioned above, the case  $u_* = 0$  corresponds to  $p = 0$  and has already been studied in the previous section. Taking  $u_* = 1$  implies that  $q = z$ , a particular case also treated in the previous section. Note that in this case, the correct solution for the critical point is not at  $u_* = 1$ , which lies on a wrong branch, but at some  $u_* < 1$  given by (4.55). The case  $u_* = \xi$  corresponds to symmetric counterparts of these particular cases. Indeed, from (4.72), we find that  $pt_* = \epsilon u_*^2$ ,  $(q-z)t_* = \epsilon\epsilon'(1-u_*)^2$  and  $(q+z)t_* = \epsilon u_*^2 + 1 - 2u_*$ , hence

$$(qt_*, zt_*) = ((1-u_*)^2, 0), (0, (1-u_*)^2), (1-2u_*, -u_*^2), (-u_*^2, 1-2u_*) \quad (4.77)$$

according to the values of the signs  $(\epsilon, \epsilon') = (+, +), (+, -), (-, +), (-, -)$ . The first two cases correspond to  $z = 0$  or  $q = 0$  and are equivalent to the  $p = 0$  particular case under the change  $p \leftrightarrow z$  or  $p \leftrightarrow q$ . The two last cases correspond to either  $p = q$  or  $p = z$ , which are now equivalent to the  $q = z$  solution. Here again,  $u_* = \xi$  lies on a wrong branch of the solution.

From now on, we shall ignore the particular cases by demanding that none of the parameters  $p, q$  and  $z$  vanishes and that they are all different. Using the  $(p, q, z)$  symmetry, we can furthermore assume without loss of generality that  $-1 \leq z/p < q/p < 1$ , with in particular  $\epsilon' = 1$ . For fixed  $q/p$  and  $z/p$  in this range, any solution of (4.74) is a candidate to describe a first order critical point, of the form (4.65), with  $t_*$  given by (4.73). We still have to make sure that both terms in (4.67) have the *same sign*, in order for the solution to read  $u_* - u \sim (1 - t/t_*)^{1/2}$ . This is the case if and only if

$$0 < u_* < 1 \quad (4.78)$$

which in turn implies  $0 < \xi < 1$ . The parametrization (4.74) is further restricted by demanding that the critical value  $u_*$  indeed corresponds to a solution  $U_1(t) = pt + O(t^2)$  for small  $t$ . Using the equation (4.44), we may solve for  $t$  as a function of  $U_1$ . We have to pick the right branch of this quadratic equation, namely that for which  $t(U_1) = \frac{U_1}{p} + O(U_1^2)$  for small  $U_1$ . It reads

$$pt(u) = \frac{u(1-u)}{(1-u)^2 - \left(\frac{q-z}{p}\right)^2 u^2} \times \left[ \left(1 - \frac{q+z}{p}\right) u(1-u) + \sqrt{u^2(1-u)^2 \left(1 - \frac{q+z}{p}\right)^2 + (1-2u) \left((1-u)^2 - \left(\frac{q-z}{p}\right)^2 u^2\right)} \right] \quad (4.79)$$

In this formulation, a critical point corresponds to a local extremum of  $pt(u)$ . The parametrization (4.74) corresponds to this branch provided

$$\epsilon = \text{sgn}(\xi - u_*) \quad (4.80)$$

This is obtained by substituting (4.74) into (4.79), and comparing the result with (4.73). So finally, we have fixed the values of  $\epsilon$  and  $\epsilon'$  in (4.74). To ensure the uniqueness of the critical point we found, we have to check that the parametrization (4.74) is bijective. We must therefore compute the Jacobian  $J$  of the map  $(u_*, \xi) \rightarrow (q/z, p/z)$  induced by (4.74): we find

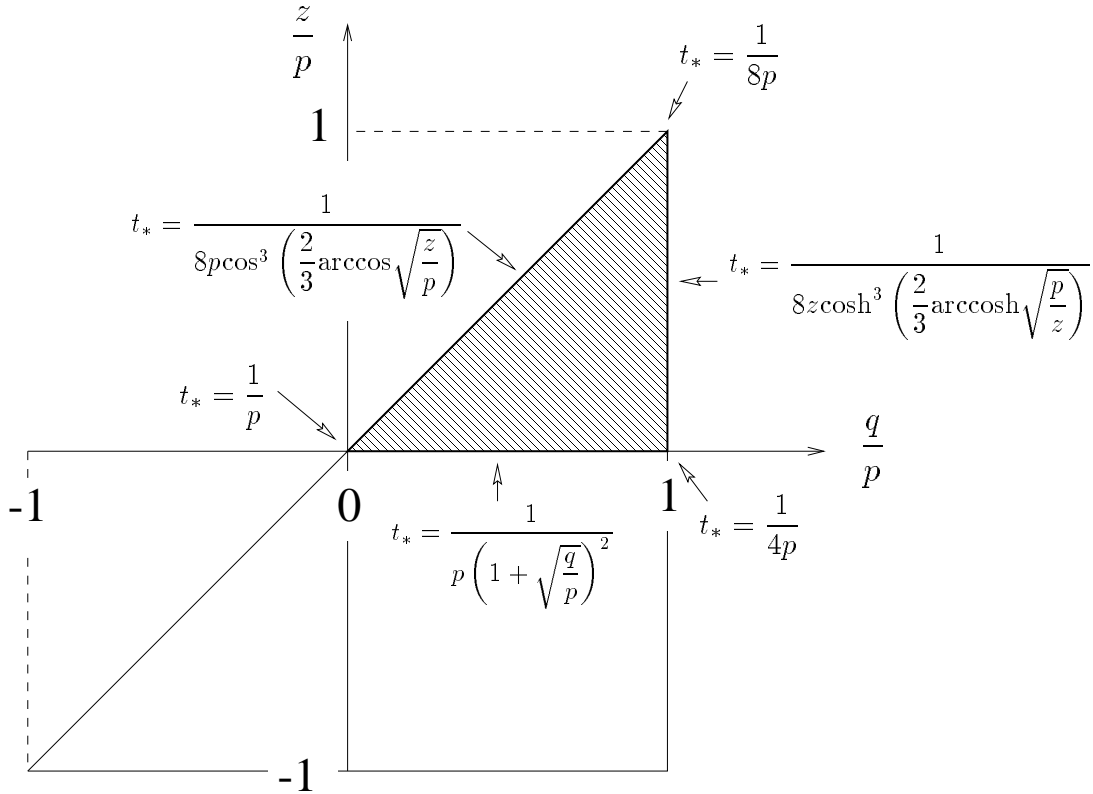
$$J = \frac{3}{16u^4\xi^2} \epsilon (u - \xi)^2 \sqrt{\frac{1-u}{1-\xi}} \quad (4.81)$$

This is non-vanishing for  $0 < u \neq \xi < 1$ . We therefore have for fixed  $q/p$  and  $z/p$  at most two acceptable solutions of (4.74), one in the range  $0 < u_* < \xi < 1$  with  $\epsilon = 1$  and one in the range  $0 < \xi < u_* < 1$  with  $\epsilon = -1$ . This latter solution can be eliminated by noting that it corresponds to a value of  $t_*$  given by (4.73) such that  $pt_* < 0$ , which is unphysical. Indeed, starting from  $pt \sim U_1$  at small  $t$ , one cannot reach continuously an extremum  $(u_*, t_*)$  of  $t(u)$  with  $u_* > 0$  and  $pt_* < 0$  without first reaching an extremum at some  $u_* > 0$ ,  $pt_* > 0$ . This fixes  $\epsilon = 1$  and  $0 < u_* < \xi < 1$ , which leaves us to at most one solution of (4.74). In this sense, the parametrization (4.73)(4.74) leads to a unique expression  $t_*(p, q, z) = (1/p)T(q/p, z/p)$  for those values of  $q/z$  and  $z/p$  in the range  $-1 \leq z/p < q/p < 1$  for which a critical point actually exists.

To determine which values of  $q/p$  and  $z/p$  give rise to a critical point, we distinguish two situations in the formula (4.79), as described in more details in Appendix B. For

$z/p < 0$ , the discriminant in (4.79) does not vanish. The function  $pt(u)$  is a strictly increasing function from  $t = 0$  at  $u = 0$  to  $t = +\infty$  at the value  $u = 1/(1 + (q - z)/p)$  where the denominator vanishes. There is no critical point in this case, hence  $R = +\infty$ . For  $z/p > 0$ , the discriminant in (4.79) vanishes at some value  $u_0$  of  $u$  strictly less than the limiting value  $1/(1 + (q - z)/p)$  for the vanishing of the denominator. At  $u_0$ , the slope of  $t(u)$  is  $-\infty$ , which implies the existence of a critical point at some  $u_*$  between 0 and  $u_0$  with  $pt_* > 0$ .

In conclusion, we find a critical point only if  $0 < z/p < q/p < 1$ , i.e. if  $p$ ,  $q$  and  $z$  have the same sign, which in turn is also the sign of  $t_*$ .



**Fig. 2:** Critical value  $t_*(p, q, z)$  in the domain  $-1 \leq z/p \leq q/p \leq 1$ . A critical point is reached only in the sub-domain  $z/p \geq 0$  (shaded region). The limiting values of  $t_*$  are explicated at the boundary of this sub-domain.

Our results are summarized in Fig.2, where we have extended the domain to  $-1 \leq z/p \leq q/p \leq 1$  by reintroducing the results of the previous section for the particular cases.

Using the results (4.76), we may finally compute the first sub-leading coefficient in the expansion

$$U_1 = u_* - \kappa(t_* - t)^{\frac{1}{2}} + \dots \quad (4.82)$$

namely

$$\kappa = \sqrt{\frac{u_*(1-u_*)}{3t_*}} \quad (4.83)$$

The free energy follows from (4.49). Actually, it is simpler to compute

$$t^2(t\partial_t)^2 f = U_1 U_2 U_3 \quad (4.84)$$

easily obtained by use of (4.42). Substituting the values (4.43) for  $U_2$  and  $U_3$ , this reads

$$\begin{aligned} t^2(t\partial_t)^2 f &= \frac{U_1}{4} \left( \left(1 - \frac{pt}{U_1}\right)^2 - \left(\frac{(q-z)t}{1-U_1}\right)^2 \right) \\ &= \frac{1}{2} \left( (1 + (p-q-z)t)U_1 - pt - U_1^2 \right) \end{aligned} \quad (4.85)$$

Substituting the expansion (4.82), we get the expansion

$$t^2(t\partial_t)^2 f = \frac{1}{4}u_*(\sqrt{\xi} - \sqrt{u_*})^2 + \frac{u_* - \xi}{4} \sqrt{\frac{u_*(1-u_*)}{3}} \sqrt{1 - \frac{t}{t_*}} + \dots \quad (4.86)$$

with a non-vanishing singular term if  $u_* \neq 0, 1, \xi$ . In particular, for  $p = q = z$ , we have  $u_* = 1/4$  and  $\xi = 1$  so that (4.86) yields

$$t^2(t\partial_t)^2 f(z, z, z, t) = \frac{1}{64} - \frac{3}{64} \sqrt{1 - 8tz} + \dots \quad (4.87)$$

in perfect agreement with (3.25)(3.26).

The above result (4.86) leads to the string susceptibility exponent  $\gamma_{str} = -1/2$ , governing the critical behavior of the free energy  $f \sim (t_* - t)^{2-\gamma_{str}}$ . This coincides with the string susceptibility exponent of pure gravity, attached to the plain enumeration of planar diagrams. This universal result holds for all the values of parameters  $(p, q, z)$  where a critical point is reached, except in the particular cases where one of these parameters is exactly zero, in which case we find the behavior (2.82). The convergence radius  $R = t_*$  may be related to the entropy  $s$  of three-coloring of the vertices of random triangulations, namely  $R = e^s$ . Actually, as a consequence of (4.86), the contribution to the free energy with a fixed number of black-white edges  $n$  ( $= 1/2 \times$  the number of triangular faces) behaves for large  $n$  as

$$\omega_n(p, q, z) \sim \frac{e^{ns(p, q, z)}}{n^{\frac{7}{2}}} \quad (4.88)$$

where the three-coloring entropy per black-white edge reads

$$s(p, q, z) = -\text{Log } t_*(p, q, z) \quad (4.89)$$

with  $t_*$  as in (4.73).



#### 4.6. Double Scaling Limit

We may now reconsider the all-genus expansion of the free energy (2.60)

$$f(p, q, z; t; N) = f_n(p, q, 1/t, N) = \sum_{h \geq 0} N^{2-2h} \varphi_h(p, q, z; t) \quad (4.90)$$

where the genus  $h$  free energy  $\varphi_h$  is a quasi-homogeneous function, satisfying  $\varphi_h(\lambda p, \lambda q, \lambda z; t/\lambda) = \lambda^{2-2h} \varphi_h(p, q, z; t)$ . When  $t$  approaches the critical point  $t_*(p, q, z)$  of (4.73), the leading singular part of  $\varphi_0$  behaves as  $(t_* - t)^{5/2}$ , and has a prefactor  $N^2$ . This suggests to perform the double-scaling limit  $t \rightarrow t_*$ ,  $N \rightarrow \infty$ , by keeping the parameter

$$y = N^{4/5} \frac{t_* - t}{t_*} \quad (4.91)$$

fixed. Let us use the expansion parameter  $\epsilon = 1/N$ . At the vicinity of  $t_*$ , let us write the total free energy

$$\begin{aligned} f(p, q, z; t; N) &= \epsilon^{-2} g_0(p, q, z) + \epsilon^{-6/5} g_1(p, q, z) y + \epsilon^{-2/5} g_2(p, q, z) y^2 \\ &+ \epsilon^0 f(y; p, q, z) + \epsilon^{2/5} g_3(p, q, z) y^3 + O(\epsilon^{4/5}) \end{aligned} \quad (4.92)$$

This defines the scaling function  $f(y; p, q, z)$ , capturing the contributions to the free energy from all genera in the double scaling limit (4.91). Its genus zero ( $y \rightarrow \infty$ ) contribution reads

$$f(y; p, q, z) = \frac{u_* - \xi}{4} \sqrt{\frac{u_*(1 - u_*)}{3}} y^{\frac{5}{2}} + \dots \quad (4.93)$$

as a consequence of (4.86). To derive an equation for  $f(y; p, q, z)$ , let us start from the recursion relation (2.52), and expand it in powers of  $\epsilon$ . With  $f_n = \text{Log } Z_n = f(p, q, z; t; N) \equiv f(z, t)$ , we find

$$\begin{aligned} f(z + \epsilon, t) + f(z - \epsilon, t) - 2f(z, t) &= \text{Log} \left( 1 + \epsilon^{-2} \frac{t}{z} (t \partial_t)^2 f(z, t) \right) \\ &= \epsilon^2 \partial_z^2 f + \frac{1}{12} \epsilon^4 \partial_z^4 f + \dots \end{aligned} \quad (4.94)$$

Substituting the expansion (4.92) for  $f$ , we finally get at the order  $\epsilon^{4/5}$  the equation

$$\frac{1}{6} \partial_y^4 f + (\partial_y^2 f)^2 = k^2 y$$

(4.95)

where  $k$  is a function of  $p, q, z$  only, fixed by the genus zero behavior (4.93) to be

$$k = \frac{5}{4}(u_* - \xi)\sqrt{3u_*(1 - u_*)} \quad (4.96)$$

Eq.(4.95) is nothing but the Painlevé I equation of two-dimensional pure quantum gravity, for the string susceptibility  $\partial_y^2 f$ . The dependence on the parameters  $p, q, z$  is entirely contained in the function  $k$ . Note that  $k$  is symmetric in  $p, q, z$ , as

$$k^2 = 75u_*^{(1)}u_*^{(2)}u_*^{(3)}(1 - u_*^{(1)})(1 - u_*^{(2)})(1 - u_*^{(3)}) \quad (4.97)$$

where  $u_*^{(1)} = u_*(p, q, z)$ ,  $u_*^{(2)} = u_*(q, p, z)$ ,  $u_*^{(3)} = u_*(z, q, p)$  denote respectively the critical values of the functions  $U_1, U_2, U_3$  of (4.42). (Recall that we have written  $u_*$  as an implicit function of  $p, q, z$  through the parametrizations (4.74), which must be inverted.)

We conclude that although our model involves more than the mere enumeration of random surfaces, its double-scaling limit satisfies the same universal Painlevé I equation as pure gravity, up to a simple rescaling involving the function  $k(p, q, z)$ .

## 5. Conclusion

In this paper, we have investigated the partition function (2.64) of the vertex tri-coloring problem for random triangulations of arbitrary genus. In addition to the exact evaluation of the tri-coloring entropy (4.89)(4.73), we have shown that the model lies in the universality class of pure two-dimensional quantum gravity, with a string susceptibility exponent  $\gamma_{str} = -1/2$ .

### 5.1. Folding of random triangulations

It is interesting to note that, in the vertex tri-coloring problem, the coloring itself does not give rise to any degree of freedom since for a given triangulation, there is only one way to color its vertices (up to global permutations of the three colors). Varying  $p, q$  and  $z$  only amounts to favoring one of these global permutations. This situation is to be contrasted with the tri-coloring of the edges of triangulations. There, even for the regular triangular lattice, the number of tri-colorings increases exponentially with the system size.

Therefore, in our problem of vertex tri-coloring, all the entropy comes from the sum over triangulations itself, as for pure gravity. The novelty here comes from the fact that this sum is restricted to *tri-colorable triangulations*. In particular, this tri-colorability

constraint implies that all vertices have an even number of surrounding triangles. This requirement is clear since the vertices around a vertex of a given color are bi-colored with alternating colors. Conversely, this parity condition turns out to be a sufficient condition to ensure tri-colorability when applied to triangulations with genus zero, as discussed in [17]. This is not the case however for triangulations with higher genus. For  $p = q = z$  where we do not distinguish between the different colors, we have found that the sum over connected planar tri-colorable triangulations obeys a quadratic equation and can be given in terms of the generating function for Catalan numbers. This result is somewhat surprising since Catalan numbers are usually associated with the enumeration of objects which can be simply decomposed into two similar smaller objects (like trees, arches, parentheses). We do not see any such decomposition here.

It is interesting to reinterpret our results in the language of folding of the triangulations. If we imagine a triangulation made of equilateral triangles with unit side length, we may want to map its vertices onto those of the regular triangular lattice in the plane, with positions  $a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ , ( $a, b$  and  $c$  integers) where  $\{\vec{e}_i\}$  is a set of three unit vectors in the plane satisfying  $\vec{e}_1 + \vec{e}_2 + \vec{e}_3 = \vec{0}$ . Such a mapping will be called a folding of the triangulation if it sends nearest neighbors of the triangulation onto nearest neighbors on the lattice, thus preserving the equilateral nature of the triangles. Two neighboring triangles will be either side by side in the plane or on top of each other with a fold between them. All triangulations cannot be folded onto the regular planar triangular lattice. The possibility for a triangulation to be folded is precisely equivalent to the tri-colorability of its vertices. To see this equivalence, we note that the regular triangular lattice can be decomposed into three regular triangular sublattices such that any neighboring vertices belong to two different sublattices. If we color each of these sublattices with three different colors, this coloring induces a tri-coloring of the vertices of any folded triangulation, which is thus tri-colorable. Conversely, if the triangulation is tri-colored, it can clearly be mapped onto a single triangle, by sending all vertices of the same color onto the same vertex of the triangle. We see here that the tri-coloring of the vertices can be viewed as a *particular* folded configuration of the triangulation where all triangles are mapped onto the same triangle. We call this configuration the *complete folding* since all edges of the lattices sustain a fold. Our result simply counts completely folded random triangulations with a different weight to the three different image vertices. Clearly, the complete folding is only a particular folded state and there are in general many different ways to fold a given tri-colorable triangulation, with an image covering in general several triangles in the plane.

This corresponds to an additional degree of freedom which is nothing but the tri-coloring of the edges of the triangulation. Indeed, any edge of the triangulation is sent onto one of the three vectors  $\vec{e}_1$ ,  $\vec{e}_2$  or  $\vec{e}_3$  and we can assign a color to this edge accordingly. In conclusion, the problem of folding of random triangulations can be identified with that of tri-coloring of *both* its vertices and its edges, a problem yet to be solved.

## 5.2. Hirota equation for multi-matrix models

The general solutions we found take the form of either some recursion relations (2.52), (3.9) or an explicit multiple sum over strictly increasing sequences of integers (4.9). The latter is a discrete analogue of the initial matrix integral itself, as it involves the square of the Vandermonde determinant of the integers summed over. Actually the large  $N$  limit of this sum is dominated by the integral (4.20), which resembles the large  $N$  approximation to the  $n \times n$  *one-matrix* integral

$$\int dM e^{-N U(M;p,q,z)} \quad (5.1)$$

where the potential  $U$  is derived from the effective action  $S$  of (4.21), namely  $U(M;p,q,z) = NS(\lambda_1, \dots, \lambda_n; p, q, z)$ , where  $\lambda_i$  denote the eigenvalues of  $M$ . This resemblance is however not quite an identity, as the corresponding eigenvalue integral should extend over the whole real line for (5.1), whereas it extends only over  $[0, \infty[$  in (4.20).

The existence of a discrete Hirota bilinear equation for  $Z_n$  is actually generic for the following multi-matrix integral

$$Z = \int dM_1 dM_2 \dots dM_k \det(1 - M_1)^{-a} \det(1 - M_k)^{-b} e^{-N \text{Tr} V(M_1, \dots, M_k)} \quad (5.2)$$

over  $n \times n$  Hermitian matrices, with the potential

$$V(M_1, \dots, M_k) = \sum_{i=1}^k V_i(M_i) + \sum_{j=1}^{k-1} u_j M_j M_{j+1} \quad (5.3)$$

with arbitrary potentials  $V_i$  (including possible logarithmic pieces). Indeed, using the Itzykson-Zuber integral (2.1), we may reduce  $Z$  to an integral over the eigenvalues  $m_j^{(i)}$  of the  $M_i$ 's. Proceeding exactly as in Sect.4.1, we may recast (5.2) into the determinant

$$\phi_n = \det \left[ \int dx_1 dx_2 \dots dx_k (1 - x_1)^{i-a-1} (1 - x_k)^{j-b-1} e^{-NV(x_1, \dots, x_k)} \right]_{1 \leq i, j \leq n} \quad (5.4)$$

up to an unimportant multiplicative constant. This is clearly a solution of the Hirota equation (3.7). This solution is singled out by its initial value  $\phi_1$ , which is a function of the details of the potential  $V$  (and in particular of the parameters  $u_1, u_2, \dots, u_{k-1}$ ). The case  $k = 1$  of a one-matrix model is also instructive. In that case, we find that [12]

$$\phi_n = \det \left[ \int dm (1-m)^{i+j-a-2} e^{-NV(m)} \right]_{1 \leq i, j \leq n} \quad (5.5)$$

satisfies the following bilinear equation

$$\boxed{\phi_{n+1}(a+2)\phi_{n-1}(a) = \phi_n(a+2)\phi_n(a) - (\phi_n(a+1))^2} \quad (5.6)$$

In this paper, we have seen however the limits of the use of the Hirota equation, insofar as the large  $N$  limit, for instance, is concerned. Because the Hirota equation is only a multi-dimensional recursion relation, strongly relying on its initial conditions, it makes it quite difficult to extract any asymptotic result. We have had to resort to a different approach to reach that goal. Perhaps a suitable mixing of the orthogonal polynomial solution and of the Hirota equation could lead to some more definite results: this is still an open question.

Finally, let us mention that our solution has displayed striking connections to the arches and meanders enumeration problems and their reexpressions within the framework of the Temperley-Lieb algebra [15][16]. We intend to return to these aspects in a later publication.

## Acknowledgements

We thank V. Kazakov and P. Wiegmann for illuminating discussions. This work was partly supported by the NSF grant PHY-9722060 (P.D.F.) and by the TMR Network contract ERBFMRXCT 960012 (B.E.).

## Appendix A. The planar free energy for small $z$

In this appendix, we compute the leading coefficient  $\varphi_1(p, q, t)$  of the free energy  $f(z; p, q, t)$  when  $z$  is small. With  $\alpha^*$  the solution (2.76) of the saddle-point equation (2.75), we have

$$\varphi_1(p, q, t) = S(\alpha^*) = p \operatorname{Log}\left(1 + \frac{\alpha^*}{p}\right) + q \operatorname{Log}\left(1 + \frac{\alpha^*}{q}\right) - \alpha^* \quad (\text{A.1})$$

Let us compute  $\psi(p, q, t) \equiv t \partial_t \varphi_1(p, q, t)$  in terms of  $\alpha^*$ . Using the formula (2.74) for  $S(\alpha)$  and the fact that, at the saddle point  $\alpha^*$ , we simply have to take the *explicit* derivative of  $S(\alpha)$  wrt  $t$ , we obtain immediately

$$\psi(p, q, t) = \alpha^* \quad (\text{A.2})$$

with  $\alpha^*$  given by (2.76). Hence, introducing the reduced variables  $u = pt$  and  $v = qt$ , we have  $\psi(p, q, t) = I(u, v)/t$ , where the function  $I(u, v)$  reads

$$I(u, v) = \frac{1}{2} (1 - u - v - \sqrt{1 - 2(u + v) + (u - v)^2}) \quad (\text{A.3})$$

and satisfies the quadratic equation

$$(I + u)(I + v) - I = 0 \quad (\text{A.4})$$

Expanding (A.3) in powers of  $u$  and  $v$ , we find

$$I(u, v) = \sum_{k \geq 1} \sum_{j=1}^k I_{k,j} u^j v^{k+1-j} \quad (\text{A.5})$$

with the integer coefficients

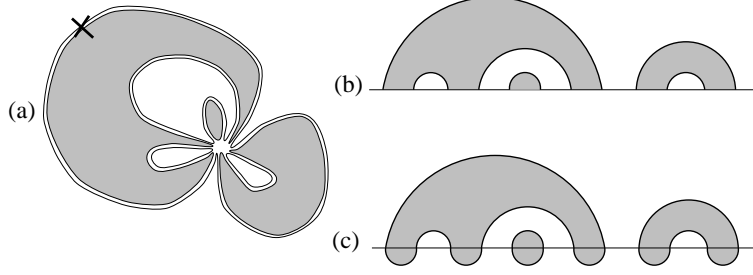
$$I_{k,j} = \frac{1}{k} \binom{k}{j} \binom{k}{j-1} \quad (\text{A.6})$$

Integrating  $\psi(p, q, t)/t = I(pt, qt)/t^2$  once wrt  $t$ , we finally get

$$\varphi_1(p, q, t) = \sum_{k \geq 1} t^k \mu_k(p, q) \quad (\text{A.7})$$

where

$$\mu_k(p, q) = \frac{1}{k} \sum_{j=1}^k I_{k,j} p^j q^{k+1-j} \quad (\text{A.8})$$



**Fig. 3:** Equivalence between (a) a bi-colored fatgraph with a unique vertex and a marked edge, (b) a system of bi-colored arches and (c) a system of arches closed into a set of connected circuits

and the result (2.80) follows. This result can be checked directly on (2.84).

From the behavior  $\alpha^* \sim (t_* - t)^{1/2}$  read on (2.76), we deduce by integrating (A.2) that  $\varphi_1(p, q, t) \sim (t_* - t)^{3/2}$ .

The numbers  $I_{k,j}$  (A.5) have already emerged in [15] in the context of arch configurations. The connection between  $\psi(p, q, t)$  and arch configurations can be stated as follows. Since  $\varphi_1(p, q, t)$  keeps only the linear term in  $z$  of  $f(z, p, q, t)$ , it is a generating function for planar (dual) black and white fatgraphs with exactly 1 vertex. The operation  $t\partial_t$  simply corresponds to specifying an edge among the  $E(\Gamma)$  edges of a graph  $\Gamma$ . We can now stretch the unique vertex of the graph into a straight line and let the edges of the graph form a system of  $E(\Gamma)$  arches drawn on top of this line (see Fig.3). The stretching operation is completely specified if we impose that the marked edge becomes the leftmost arch, with its white adjacent face above it. The function  $\psi(p, q, t)$  is thus a generating function for systems of bi-colored arches. Note that the same arch configuration is obtained exactly  $|\text{Aut}(\Gamma)|$  times from the different markings of a given graph  $\Gamma$ , which in turn cancels the symmetry factor in the diagrammatic expansion. We thus have

$$\psi(p, q, t) = \sum_{k \geq 1} \sum_{j=1}^k M_{k,j} t^k p^j q^{k+1-j} \quad (\text{A.9})$$

where  $M_{k,j}$  is the number of distinct configurations of  $k$  arches drawn on top of an oriented line, separating  $k+1$  domains in exactly  $j$  black domains and  $\ell = k+1-j$  white domains when colored in black and white with alternating colors, the outermost domain being white.

The number  $M_{k,j}$  has been computed in [15], with the result

$$M_{k,j} = I_{k,j} \quad (\text{A.10})$$

given in (A.5), with a slightly different interpretation as the number of systems of  $k$  arches drawn on top of a line which give rise to exactly  $j$  connected circuits when closed below the line by elementary arches connecting *successive* points on the line in pairs. This number is clearly the same as the number of bi-colored systems with  $j$  black domains and  $k + 1 - j$  white domains by coloring in black the interior of the  $j$  connected circuits (see Fig.3).

The interpretation of  $\psi(p, q, t)$  as the generating function for bi-colored systems of arches allows for a more direct computation by use of a simple recursive relation for the  $M_{k,j}$ . Denoting by

$$M_k(p, q) = \sum_{j=1}^k M_{k,j} p^j q^{k+1-j} \quad (\text{A.11})$$

we have the recursion relation

$$M_{k+1}(p, q) = \sum_{j=0}^k M_j(q, p) M_{k-j}(p, q) \quad (\text{A.12})$$

with the convention  $M_0(p, q) = q$ . This equation simply expresses the fact that the leftmost arch separates the system of bi-colored arches into two systems of bi-colored arches. Note that  $p$  and  $q$  are exchanged in one of these two systems of arches. The above recursion relation translates into a quadratic equation for  $\psi(p, q, t) = \sum_{k \geq 1} M_k(p, q) t^k$

$$\psi(p, q, t) = t(\psi(q, p, t) + p)(\psi(p, q, t) + q) \quad (\text{A.13})$$

which, since  $\psi(p, q, t)$  is by definition symmetric in the exchange  $p \leftrightarrow q$ , is nothing but the equation (A.4).

Note finally that when  $p = q$ ,  $I(u, u) = u(C(u) - 1)$ , where  $C(u)$  is the generating function of the Catalan numbers (3.22)(3.23), and it follows that

$$\mu_k(p, p) = p^{k+1} \frac{c_k}{k} = 2p^{k+1} \frac{(2k-1)!}{k!(k+1)!} \quad (\text{A.14})$$

## Appendix B. Existence of a critical point

We start from the solution  $t_{\pm}(u)$  of the equation  $P(u, t(u)) = 0$

$$\begin{aligned} pt_{\pm}(u) &= \frac{u(1-u)}{(1-u)^2 - \left(\frac{q-z}{p}\right)^2 u^2} \times \left[ \left(1 - \frac{q+z}{p}\right) u(1-u) \right. \\ &\quad \left. \pm \sqrt{u^2(1-u)^2 \left(1 - \frac{q+z}{p}\right)^2 + (1-2u) \left((1-u)^2 - \left(\frac{q-z}{p}\right)^2 u^2\right)} \right] \end{aligned} \quad (\text{B.1})$$



We want to know whether or not the branch  $t_+(u)$  (satisfying  $pt_+(u) \sim u$  at small  $u$ ) reaches continuously by increasing  $u$  from 0 a maximum at some value  $u_*$  between 0 and 1. We assume here that  $-1 \leq z/p < q/p < 1$ .

We first note that the denominator in (B.1) vanishes for  $u = u_1$  with

$$u_1 = \frac{1}{1 + \frac{q-z}{p}} \quad (\text{B.2})$$

between  $1/3$  and  $1$ . Let us study the possible vanishing of the discriminant in (B.1). This discriminant vanishes for

$$\left(1 - \frac{q+z}{p}\right)^2 = \left(\frac{1}{(1-u)^2} - \frac{1}{u^2}\right) \left((1-u)^2 - \left(\frac{q-z}{p}\right)^2 u^2\right) \quad (\text{B.3})$$

In the domain  $0 < u < 1$ , the r.h.s. is positive for  $u$  between  $1/2$  and  $u_1$ , and is maximum at

$$u_2 = \frac{1}{1 + \sqrt{\frac{q-z}{p}}} \quad (\text{B.4})$$

with the value

$$\left(1 - \frac{q-z}{p}\right)^2 \quad (\text{B.5})$$

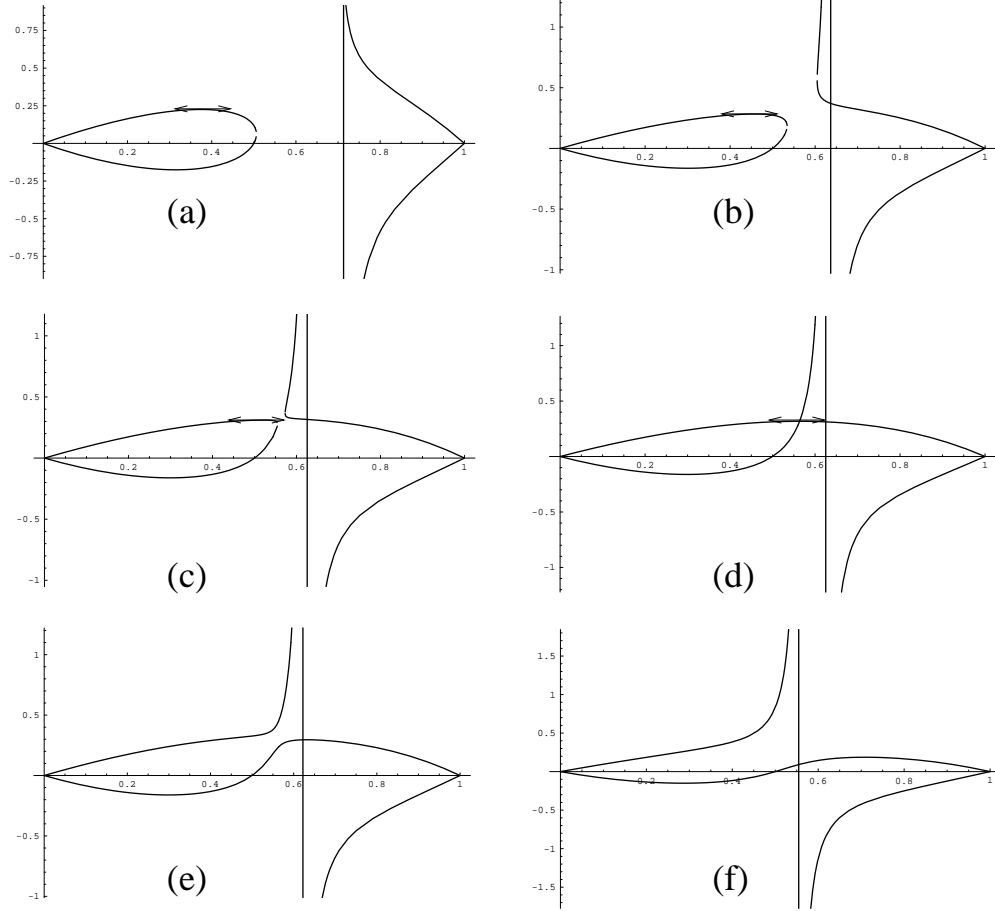
The discriminant will thus vanish if and only if

$$\left(1 - \frac{q+z}{p}\right)^2 \leq \left(1 - \frac{q-z}{p}\right)^2 \quad (\text{B.6})$$

This is true in the domain  $-1 \leq z/p < q/p < 1$  if and only if  $z/p \geq 0$ . In this case, the discriminant vanishes exactly once in the range  $0 < u \leq u_2$  for some value  $u_0$  with moreover  $1/2 \leq u_0 \leq u_2 < u_1$  (since  $(q-z)/p < 1$ ). At  $u = u_0$ , the two branches  $t_{\pm}(u)$  meet with an infinite slope (if  $z/p > 0$ ) and make a loop. Clearly, the branch  $t_+(u)$  must have a maximum at some  $u_* < u_0$  and we find a critical point. For  $z/p = 0$ , we find a critical point at  $u_* = u_0 = u_2$ .

Conversely, if  $z/p < 0$ , the discriminant does not vanish in the range  $0 < u < 1$  and, since  $(1 - (q+z)/p) > 0$ , we find that  $t_+(u)$  is a strictly increasing function which diverges at  $u = u_1$ . We find no maximum in this case, hence no critical point.

We give an example of these behaviors in the Figure 4.



**Fig. 4:** The behavior of  $pt_{\pm}(u)$  for  $0 \leq u \leq 1$ , here represented for  $q/p = 2/3$  and decreasing values of  $z/p$ . A critical point corresponds to a maximum of  $pt(u)$  lying on the *correct* branch of solution, i.e. that leaving the origin with a positive slope. For  $z/p > 0$  (fig. a,b,c) this branch forms a loop with the *wrong* branch leaving the origin with a negative slope: a maximum is reached on the correct branch before the two branches meet. For  $z/p = 0$  (fig. d), a maximum is also reached, lying now exactly at the meeting point of the two branches. For  $z/p < 0$  (fig. e,f), the two branches avoid each other and no maximum is found for the correct branch.

## References

- [1] R. Baxter, *Exactly Solved Models of Statistical Mechanics*, Oxford (1982).
- [2] R. Baxter, J. Math. Phys. **11** (1970) 784; J. Phys. **A19** (1986) 2821.
- [3] P. Di Francesco and E. Guitter, Europhys. Lett. **26** (1994) 455, cond-mat/9406041.
- [4] G. Cicuta, L. Molinari and E. Montaldi, Phys. Lett. **B306** (1993) 245.
- [5] L. Chekhov and C. Kristjansen, Nucl. Phys. **B479** (1996) 683, hep-th/9605013.
- [6] C. Kristjansen and B. Eynard, Durham preprint DTP-97/55 , cond-mat/9710199.
- [7] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *2D Gravity and Random Matrices*, Physics Reports **254** (1995) 1, hep-th/9306153.
- [8] R.C. Penner, Comm. Math. Phys. **113** (1987) 299; J. Diff. Geom. **27** (1988) 35.
- [9] Yu. Makeenko, Phys. Lett. **B314** (1993) 197, hep-th/9306043.
- [10] V.A. Kazakov and A.A. Migdal, Nucl. Phys **B397** (1993) 214.
- [11] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21** (1980) 411.
- [12] V. Kazakov, private communication.
- [13] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, Comm. Math. Phys. **188** (1997) 267, hep-th/9604080.
- [14] W. Tutte, Canad. Jour. of Math. **15** (1963) 249.
- [15] P. Di Francesco, O. Golinelli and E. Guitter, Mathematical and Computer Modeling **144** (1997), hep-th/9506030.
- [16] P. Di Francesco, O. Golinelli and E. Guitter, Comm. Math. Phys. **186** (1997) 1, hep-th/9602025.
- [17] this problem is discussed in the mathematical entertainments section of the Mathematical Intelligencer Volume 19 number 4 (1997) 48.